AdS/CFT Correspondence
with applications to condensed matter

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I. Quantum Field Theory and Gauge Theory

Convention: $\hbar = 1 = c$ Metric signature: $(- + + +)$

Action:

$$S = \int d^4x \, \mathcal{L}\,(\phi, \partial_\mu \phi)$$

Classical equations of motion:

$$\frac{\delta S}{\delta \phi} = 0 \quad \longrightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

Time-ordered vacuum expectation values by path integrals:

$$\langle 0| \bullet |0\rangle = \int \mathcal{D}\phi e^{iS[\phi]} \bullet$$

$$\mathcal{D}\phi = \prod_i \int d\phi(x_i)$$

n-point correlation function:

$$G_n(x_1, \ldots, x_n) = \langle 0 | T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\} | 0 \rangle$$

obtained from generating functional:

$$Z[J] = \int D\phi e^{iS[\phi] + i\int d^4x J(x)\phi(x)}$$

Euclidean version via Wick rotation:

Minkowskian signature $(-+++)$ Euclidean signature $(++++)$

$$t = -it_E\,, \qquad iS = -S_E$$

Thus via analytical continuation in the path integral

$$Z[J] = Z_E[J] = \int D\phi e^{-S_E[\phi] + \int d^4x J(x)\phi(x)} = {}_J\langle 0|0\rangle_J$$

Correlation function:

$$G_n(x_1,\ldots,x_n) = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z_E[J]\Big|_{J=0}$$

Example of simple QFT:

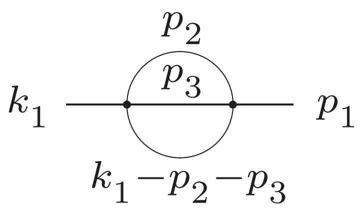
$$S_E = \int d^4x \Big[\tfrac{1}{2} \underbrace{(\partial_\mu \phi)^2}_{\left(\frac{\partial\phi}{\partial t_E}\right)^2 + \left(\vec{\nabla}\phi\right)^2} + \tfrac{1}{2} m^2 \phi^2 + \underbrace{V(\phi)}_{\text{e.g. } \lambda\phi^4} \Big]$$

Feynman rules:

1. propagator: —— $= \Delta(x,y) = \dfrac{1}{-\partial_\mu \partial^\mu + m^2} = \displaystyle\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + m^2 - i\epsilon}$

2. vertex: • $= \int d^4x \, (-\lambda)$

e.g. setting sun diagram: m^2 means $m^2 - i\epsilon$ in the following



$$= \int \frac{d^4 p_2}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \; \frac{1}{p_2^2 + m^2} \; \frac{1}{p_3^2 + m^2} \; \frac{\lambda^2 (2\pi)^4 \delta^4(p_1 - k_1)}{(k_1 - p_2 - p_3)^2 + m^2}$$

Extraction of S-matrix elements (in Minkowski space) from Fourier-transformed $n+m$ point correlation functions in Minkowski space:

$$\tilde{G}_{n+m}(p_1, \ldots, p_n; k_1, \ldots, k_m) \sim$$

$$\left(\prod_{i=1}^{n} \frac{\sqrt{Z} i}{p_i^2 + m^2} \right) \left(\prod_{j=1}^{m} \frac{\sqrt{Z} i}{k_j^2 + m^2} \right) \langle p_1, .., p_n | S | k_1, .., k_m \rangle$$

Correlation function of composite fields $\mathcal{O}$: e.g. $\mathcal{O}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi$ add

$$\int d^4 x J_{\mu\nu} \mathcal{O}_{\mu\nu} \quad \text{to} \quad \mathcal{L}$$

and proceed as before.

OK, lets move on to gauge-fields now!

Electromagnetic field:

$$A_\mu = (-\phi, \vec{A}) \qquad \text{1 Form} \qquad A = A_\mu dx^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{2 Form} \qquad F = dA$$

$$= 2\partial_{[\mu} A_{\nu]} \qquad = F_{|\mu\nu|} dx^\mu \wedge dx^\nu$$

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

e.g. $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \quad E_1 = \partial_1 A_0 - \partial_0 A_1 = -F_{01}$

$\vec{B} = \vec{\nabla} \times \vec{A}, \quad B_1 = \partial_2 A_3 - \partial_3 A_2 = F_{23}$

gauge invariance $\quad \delta A_\mu = \partial_\mu \lambda \quad \Rightarrow \quad F_{\mu\nu}$ gauge invariant

Classically A is not observable (not gauge-invaraint), only $F = dA$ is.

$$S_{\text{Mink.}} = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}$$

$$S_{\text{Euclid.}} = \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}$$

Coupling to spinor and scalar:

Add

$$\mathcal{L}_\psi = -\,\bar{\psi}(\not{D} + m)\psi \quad \text{with} \quad \bar{\psi} = \psi^\dagger i\gamma^0,\ \ \not{D} = \gamma^\mu D_\mu\,,\ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\mathcal{L}_\phi \;=\; -(D_\mu\phi)D^\mu\phi \quad \text{with} \quad D_\mu = \partial_\mu - ieA_\mu$$

Explicit representation of gamma matrices: σ, τ standard Pauli matrices

$$\gamma^0 = I \otimes \tau_3, \quad \gamma^i = \sigma^i \otimes i\tau_2 \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = I \otimes \tau_1 \quad \{\gamma^5, \gamma^\mu\} = 0$$

Differential forms convenient in electrodynamics and GRT,
they get ubiquitous in non-abelian gauge theories and string theory.

Let's use them to discuss Dirac's quantization of charge, if a magnetic monopole g exists somewhere, following the discussion in
Zee's 'QFT in a nutshell' textbook
Let magnetic monopole g sit at $\vec{x} = 0$.

Integrating the resulting $F = dA$ over a S_2 surrounding the monopole we dont get zero, as usually, but like in Gauss' law $\int_{S_2} *F =$charge

$$\int_{S_2} F = g$$

The $\vec{B}$ field of the monopole has the same form as the $\vec{E}$ field for an electric point-charge,

$$\vec{B} = \frac{g\hat{r}}{4\pi r^2}$$

Written as the 2-form F in spherical coordinates this is

$$F = -\frac{g}{4\pi} sin\theta \; d\theta \wedge d\varphi$$

$d\varphi$ is not well-defined on the north- or southpole, but F is neverthe-less, since $sin\theta$ vanishes there. But $A = \frac{g}{4\pi} cos\theta \; d\varphi$, for $F = dA$, is **not** well defined there. On the northpole our luck is corrected with $A_N = \frac{g}{4\pi}(cos\theta - 1)d\varphi$, on the southpole we need $A_S = \frac{g}{4\pi}(cos\theta + 1)d\varphi$.

In the overlap region everywhere between both poles

$$A_S = A_N + \frac{2g}{4\pi} d\varphi$$

i.e. both forms of the vector potential are gauge equivalent **classically**. **Quantally** however, only those gauge transformations of A are permitted which act trivially on the wave-function ψ.

$$A_S = A_N + \frac{1}{ie} e^{-i\Lambda} d e^{i\Lambda}$$

with $\Lambda = \frac{ge}{2\pi}\varphi$ and for the wave function ψ_N and ψ_S in both domains, $\psi_S = e^{i\Lambda}\psi_N$. So $e^{i\Lambda}$ must be 2π-periodic in φ :

$$ge = 2\pi n$$

Magnetic monopoles are extremely strongly coupled! $g^2 \sim (4\pi^2)137$

Conclusion: electromagnetic duality

- is quantum in origin
- is a weak/strong duality

This is a somewhat familiar duality and an early precursor of AdS/CFT duality, the latter being a lot stranger, however!

The electrodynamic 1-form A_μ is sourced by the current-vector $J^\mu(x)$

$$J^\mu(x) = \int d\tau \frac{dX^\mu}{d\tau} \delta^{(D)}(x - X(\tau))$$

along the worldline (wl) $X^\mu(\tau)$ via the term in the Lagrangian

$$\mathcal{L} = \int d^D x A_\mu(x) J^\mu(x) = \int_{wl} A$$

For a string there is a similar reparametrization-invariant (and thereby neccessarily antisymmetric) tensor-current $J^{\mu\nu}(x)$ along the 'world-sheet' (ws) $X^{\mu}(\sigma,\tau)$

$$J^{\mu\nu}(x) = \int d\tau d\sigma \left(\partial_\tau X^\mu \partial_\sigma X^\nu - \partial_\tau X^\nu \partial_\sigma X^\mu\right) \delta^{(D)}(x - X(\sigma,\tau))$$

which sources its own two-form gauge potential $B_{\mu\nu}(x)$ via

$$\mathcal{L} = \int d^D x B_{|\mu\nu|}(x) J^{\mu\nu}(x) = \int_{ws} B_{|\mu\nu|}(X) dX^\mu \wedge dX^\nu = \int_{ws} B.$$

We shall meet that again in a short while.

Yang-Mills fields:

generators T_R^a, $\qquad [T_R^a, T_R^b] = i f^{abc} T_R^c$

representation R, T_R^a traceless, hermitian, f^{abc} real, completely antisymmetric

adjoint representation: $(T_A^a)^{bc} = -i f^{abc}$

e.g. $SU(2): \quad f^{abc} = \epsilon_{abc} \quad , \quad T^a = \frac{1}{\hbar} J^a$ hermitian

gauge-field $(A_\mu)_{..} = A_\mu^a (T^a)_{..}$ becomes a matrix, and so (with the gauge-field c

$$D_\mu = \partial_\mu - igA_\mu, \quad D \wedge D = -igF, \quad F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = F_{\mu\nu}^a T^a$$

They are again forms:

$$\begin{aligned} A_{..} &= (A_\mu)_{..} dx^\mu \\ F_{..} &= \tfrac{1}{2}(F_{\mu\nu})_{..} dx^\mu \wedge dx^\nu \\ F_{..} &= dA_{..} - ig(A \wedge A)_{..} \\ &= dx^\mu \wedge dx^\nu (\partial_{[\mu} A_{\nu]_{..}} + \tfrac{-ig}{2}[A_\mu, A_\nu]_{..}) \end{aligned}$$

Normalization of generators:

$$\text{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$$

trace taken in the fundamental representation
(e.g. $SU(2)$ in spin-$\frac{1}{2}$ representation $\quad T^a = \frac{1}{2}\sigma^a$)
gauge transformation infinitesimal:

$$\delta A^a_\mu = (D_\mu \epsilon)^a\,, \quad \epsilon \text{ infinitesimal}$$

with

$$(D_\mu \epsilon)^a = \partial_\mu \epsilon^a - i g f^{abc} A^b_\mu \epsilon^c$$

Finite transformation:

$$A'_\mu = U^{-1}(x) A_\mu U(x) + \frac{i}{g} U^{-1}(x) \partial_\mu U(x); \quad U_{..} = (e^{i\epsilon^a T^a})_{..} = (e^{i\epsilon})_{..}$$

$$F'_{\mu\nu} = U^{-1}(x) F_{\mu\nu} U(x)$$

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig(T^a)_{ij} A^a_\mu$$

e.g. for fermion again $\bar{\psi}\partial_\mu\psi \to \bar{\psi}D_\mu\psi$

Noether current:
symmetry transformation on $\mathcal{L}$

$$\delta_{\text{symm}}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^i}\delta_{\text{symm}}\phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\partial_\mu \delta_{\text{symm}}\phi^i$$

A symmetry transformation does not neccessarily leave the Lagrangian, but always, by definition, the action invariant

$$\frac{\delta_{\text{symm}} S}{\delta_{\text{symm}}\phi^i} = \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = 0$$

even when arbitrary variations of the action will not vanish, i.e. even 'off shell'. This defining condition of symmetry transformations of actions can be used to rewrite the symmetry transformation of the Lagrangian, eliminating the term with $\frac{\partial \mathcal{L}}{\partial \phi^i}\delta_{\text{symm}}\phi^i$. Then

$$\delta_{\text{symm}}\mathcal{L} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\delta_{\text{symm}}\phi^i\right)$$

Therefore if $\delta_{\text{symm}}\mathcal{L} = \partial_\mu K^\mu$ which may of course also be 0,

$$j^\mu_{\text{symm}} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_{\text{symm}} \phi^i - K^\mu$$

is conserved. (Noether's theorem)

e. g. if

$$\delta_{\text{symm}}\phi^i = \epsilon^a T^{a,ij}\phi^j$$

then

$$\delta_{\text{symm}}\mathcal{L} = \epsilon^a\,\partial_\mu \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)} T^{a,ij}\,\phi^j}_{j^{a,\mu}}$$

or, with fermions

$$\mathcal{L}_{\text{F}} = \bar{\psi}^i(i\gamma^\mu D_\mu - m)\psi^i$$

with

$$\delta\psi^i = \epsilon^a T^{a,ij}\,\psi^j\,.$$

Hence

$$j^{a,\mu} = \bar{\psi}^i\gamma^\mu T^{a,ij}\,\psi^j\,,$$

composite operator, gauge invariant.

Anomalies:

Classical symmetry can break down in quantum theory.

Simple example: massless fermions with a U(1)-gauge field in 4 space-time dimensions

$$\mathcal{L}_\psi = \bar{\psi} i \not{D} \psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

with $D_\mu = \partial_\mu - igA_\mu$.

For massless fermions, left- and right-handed components dont mix, the vector current $j^\mu = \bar{\psi}\gamma^\mu\psi$ and the axial vector current $j^\mu_A = \bar{\psi}\gamma^\mu\gamma_5\psi$ are classically both conserved.

Quantally a 1-loop calculation, in which the fermion runs in the triangle-loop, shows

$$\partial_\mu j^\mu_A = -\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Reason for the quantal breakdown of the axial symmetry: non-invariance of the integration measure in the path integral

Adapted from **M. Srednicki's**: **Quantum Field Theory**:

The simplest example is electrodynamics with a massless Dirac field Ψ with charge $Q = +1$. The lagrangian is

$$\mathcal{L} = i\overline{\Psi}\not{D}\Psi - \tfrac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} , \tag{76.1}$$

where $\not{D} = \gamma^\mu D_\mu$ and $D_\mu = \partial_\mu - igA_\mu$. (We call the coupling constant g rather than e because we are using this theory as a formal example rather than a physical model.) We can write Ψ in terms of two left-handed Weyl fields χ and ξ via

$$\Psi = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix} , \tag{76.2}$$

where χ has charge $Q = +1$ and ξ has charge $Q = -1$. In terms of χ and ξ, the lagrangian is

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu(\partial_\mu - igA_\mu)\chi + i\xi^\dagger\bar{\sigma}^\mu(\partial_\mu + igA_\mu)\xi - \tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\ . \tag{76.3}$$

The lagrangian is invariant under a U(1) gauge transformation

$$\Psi(x) \to e^{-ig\Gamma(x)}\Psi(x)\ , \tag{76.4}$$

$$\overline{\Psi}(x) \to e^{+ig\Gamma(x)}\overline{\Psi}(x)\ , \tag{76.5}$$

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\Gamma(x)\ . \tag{76.6}$$

In terms of the Weyl fields, eqs. (76.4) and (76.5) become

$$\chi(x) \to e^{-ig\Gamma(x)}\chi(x)\ , \tag{76.7}$$

$$\xi(x) \to e^{+ig\Gamma(x)}\xi(x)\ . \tag{76.8}$$

Because the fermion field is massless, the lagrangian is also invariant under a global symmetry in which χ and ξ transform with the same phase,

$$\chi(x) \to e^{+i\alpha}\chi(x)\ , \tag{76.9}$$

$$\xi(x) \to e^{+i\alpha}\xi(x)\ . \tag{76.10}$$

In terms of Ψ, this is

$$\Psi(x) \to e^{-i\alpha\gamma_5}\Psi(x) \; , \tag{76.11}$$

$$\overline{\Psi}(x) \to \overline{\Psi}(x)e^{-i\alpha\gamma_5} \; . \tag{76.12}$$

This is called *axial U(1) symmetry*, because the associated Noether current

$$j^\mu_{\rm A}(x) \equiv \overline{\Psi}(x)\gamma^\mu\gamma_5\Psi(x) \tag{76.13}$$

is an axial vector (that is, its spatial part is odd under parity). Noether's theorem leads us to expect that this current is conserved: $\partial_\mu j^\mu_{\rm A} = 0$. However, in this section we will show that the axial current actually has an *anomalous divergence*,

$$\partial_\mu j^\mu_{\rm A} = -\frac{g^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \; . \tag{76.14}$$

Using the LSZ formula for photons

$$\langle p,q|j_{\rm A}^{\rho}(z)|0\rangle = (ig)^2\,\varepsilon_\mu\varepsilon'_\nu\int d^4x\,d^4y\,e^{-i(px+qy)}\langle 0|{\rm T}j^\mu(x)j^\nu(y)j_{\rm A}^{\rho}(z)|0\rangle\ , \tag{76.15}$$

where

$$j^\mu(x) \equiv \overline{\Psi}(x)\gamma^\mu\Psi(x) \tag{76.16}$$

is the Noether current corresponding to the U(1) gauge symmetry. Since both $j^\mu(x)$ and $j_{\rm A}^{\mu}(x)$ are Noether currents, we expect the Ward identities

$$\frac{\partial}{\partial x^\mu}\langle 0|{\rm T}j^\mu(x)j^\nu(y)j_{\rm A}^{\rho}(z)|0\rangle\ =\ 0\ , \tag{76.17}$$

$$\frac{\partial}{\partial y^\nu}\langle 0|{\rm T}j^\mu(x)j^\nu(y)j_{\rm A}^{\rho}(z)|0\rangle\ =\ 0\ , \tag{76.18}$$

$$\frac{\partial}{\partial z^\rho}\langle 0|{\rm T}j^\mu(x)j^\nu(y)j_{\rm A}^{\rho}(z)|0\rangle\ =\ 0\ , \tag{76.19}$$

to be satisfied.

If we use eq. (76.19) in eq. (76.15), we see that we expect

$$\frac{\partial}{\partial z^\rho}\langle p,q|j_{\rm A}^{\rho}(z)|0\rangle = 0\ . \tag{76.20}$$

However, our experience leads us to proceed more cautiously.

Let us define $C^{\mu\nu\rho}(p,q,r)$ via

$$(2\pi)^4\delta^4(p+q+r)C^{\mu\nu\rho}(p,q,r) \equiv \int d^4x\, d^4y\, d^4z\, e^{-i(px+qy+rz)}\langle 0|\mathrm{T}j^{\mu}(x)j^{\mu}(y)j^{\rho}_{\mathrm{A}}(z)|0\rangle\ . \qquad (76.21)$$

Then we can rewrite eq. (76.15) as

$$\langle p,q|j^{\rho}_{\mathrm{A}}(z)|0\rangle = -g^2\varepsilon_{\mu}\varepsilon'_{\nu}C^{\mu\nu\rho}(p,q,r)e^{irz}\Big|_{r=-p-q}\ . \qquad (76.22)$$

Taking the divergence of the current yields

$$\langle p,q|\partial_{\rho}j^{\rho}_{\mathrm{A}}(z)|0\rangle = -ig^2\varepsilon_{\mu}\varepsilon'_{\nu}r_{\rho}C^{\mu\nu\rho}(p,q,r)e^{irz}\Big|_{r=-p-q}\ . \qquad (76.23)$$

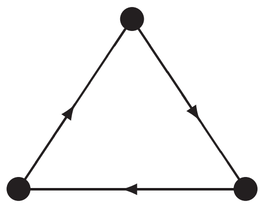
The expected Ward identities become

$$p_{\mu}C^{\mu\nu\rho}(p,q,r) = 0\ , \qquad (76.24)$$

$$q_{\nu}C^{\mu\nu\rho}(p,q,r) = 0\ , \qquad (76.25)$$

$$r_{\rho}C^{\mu\nu\rho}(p,q,r) = 0\ . \qquad (76.26)$$

$C^{\mu\nu\rho}(p,q)$ is to be calculated from the triangle diagram



with $i\gamma^\rho\gamma^5$, $i\gamma^\mu$, and $i\gamma^\nu$ on the three corners, in which the fermion with momentum k circulates

$$C^{\mu\nu\rho}(p,q) = -i^3 \int \frac{d^4k}{(2\pi)^4} \quad tr\left(\gamma^\rho\gamma^5 \frac{1}{\not k - \not r}\gamma^\nu \frac{1}{\not k - \not p}\gamma^\mu \frac{1}{\not k} + \gamma^\rho\gamma^5 \frac{1}{\not k - \not r}\gamma^\mu \frac{1}{\not k - \not p}\gamma^\nu \frac{1}{\not k}\right)$$

To check eqs. (76.24–76.26), we compute $C^{\mu\nu\rho}(p,q,r)$ with Feynman diagrams. At the one-loop level,

the three vertex factors are

γ^μ, γ^ν, and $\gamma^\rho\gamma_5$

we could choose a regularization scheme that preserved eqs. (76.24) and (76.25), but not also (76.26).

we definitely want to preserve eqs. (76.24) and (76.25), because these imply conservation of the current coupled to the gauge field, which is necessary for gauge invariance. On the other hand, we are less enamored of eq. (76.26), because it implies conservation of the current for a mere global symmetry.

Using our results from section 75, we find that preserving eqs. (76.24) and (76.25) results in

$$r_\rho C^{\mu\nu\rho}(p,q,r) = -\frac{i}{2\pi^2}\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta + O(g^2) \qquad (76.28)$$

Consequences of the anomaly we discussed:

- With A^μ the electromagnetic 4-potential: classical $\partial_\mu j_A^\mu = 0$
  but quantum $\partial_\mu j_A^\mu = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$
  so $\partial_\mu j_A^\mu$ has become a sink creating two photons.

- The total em current j^μ is free of anomalies. Written as sum

  $$j^\mu = j_R^\mu + j_L^\mu$$

  of right-handed and left-handed currents

  $$j_{R/L}^\mu = \bar{\psi}\gamma^\mu \frac{1 \pm \gamma_5}{2}\psi \quad \text{we get}$$

  $$j_A^\mu = j_R^\mu - j_L^\mu$$

  Each of the two components carries, with opposite signs, half of the total anomaly.

- If the fermion has a nonvanishing mass, the

  symmetry $\psi \to e^{i\theta\gamma^5}\psi$ is lost: classically $\partial_\mu j_A^\mu = 2m\bar{\psi}i\gamma^5\psi$.

  This gets a mass-independent quantum correction of the calculated size of the anomaly.

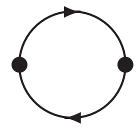
- For a nonabelian gauge field the same calculation carries through but $A = A^a T^a$ and $F = F^a T^a$ become matrix 1- and 2-forms. Result for the anomaly now is: $\partial_\mu j_A^\mu = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} tr F_{\mu\nu}F_{\rho\sigma}$.

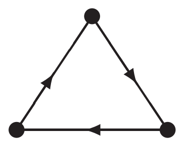
  F now contains terms linear and quadratic in A. So the sink of j_A^μ is now due to creation-processes of 2, 3, or 4 gauge bosons.

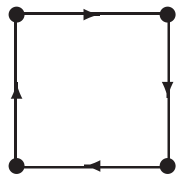
- **Adler and Bardeen** proved that the anomaly gets no contributions from higher order diagrams.

Some general rules for anomalies:

1. occur only in 1-loop order and given by polygon graphs

2. in $d = 2$: only 

3. in $d = 4$: only 

4. in $d = 6$: only 

Summary:

- Correlation functions from Feynman diagrams represented as derivatives of partition function

- S-matrix as coefficients of poles of correlations functions (LSZ)

$$\tilde{G}_{n+m} \sim \prod_{i,j} \frac{\sqrt{Z}\, i}{k_i^2 + m_i^2 - i\epsilon} \frac{\sqrt{Z}\, i}{p_j^2 + m_j^2 - i\epsilon} \left\langle \{p_i\} \middle| S \middle| \{k_j\} \right\rangle$$

- Minimal coupling generalizes also to non-abelian gauge theories

- Field-strength is nonlinear in gauge-vector potential in non-abelian case

- Hence a self-coupling of non-abelian gauge fields arises

- Noether theorem leads to conserved currents for global symmetries
- May have anomalies in quantum case
- Anomalies occur only in 1-loop Feynman diagrams
  in $d = 4$ only from triangle diagrams, i.e. in 3-point functions
- In gauge theory the current due to a global symmetry
  is also gauge invariant

For an excellent modern reference on QFT see:
Mark Srednicki, 'Quantum Field Theory',
Cambridge University Press 2007; see in particular pp 407–526;
(online version available via Google).