AdS/CFT Correspondence with Applications to Condensed Matter

Robert Graham

UNIVERSITÄT DUISBURG ESSEN

SFB/TR 12: Symmetries and Universality in Mesoscopic Systems

I. Quantum Field Theory and Gauge Theory

II. Conformal Field Theory

III. A brief Introduction to Supersymmetry

A little bit of 'Susy'

(largely following McGreevy, 8.821 F008 Lectures 5, 6)

Physical reasons for susy:

- it stabilizes the mass of the Higgs against radiative corrections quadratic in the large-energy cutoff, since susy would reduce this scale to the one below which susy is spontaneously broken; above this scale the mass of the Higgs is tied to that of its fermionic partner, which receives only logarithmic radiative corrections ;

- it changes the trajectories of RG flow such that the gauge couplings may unify at very high energies: $E_{GUT} \simeq 10^{15} GeV$

- it reduces the theoretical error in the cosmological constant from $\sim 10^{120}$ to $\sim 10^{60}$.

Susy is a **fermionic extension** of the
Poincare group,

$$P^{\mu} \quad , \quad M_{\mu\nu}$$

with group submanifolds $\quad R_{1,3} \quad , \quad SO(1,3) \sim SU(2) \times SU(2)$

which has only bosonic generators and is **not extendable by any symmetry-group with bosonic generators** (theorem of Coleman and Mandula)

Splitting Lorentz to rotations and boosts $\quad J^i = \epsilon^{ijk} M_{jk} \quad , \quad K^i = M^{0i}$

Separating into two decoupled $SU(2)$ algebras $\quad A^i_{(B^i)} = (J^i +_{(-)} iK^i)/\sqrt{2}$

with

$$\left[A^i, A^j\right] = i\epsilon^{ijk} A_k, \qquad \left[B^i, B^j\right] = i\epsilon^{ijk} B_k, \qquad \left[A^i, B^j\right] = 0$$

Representations of $SO(1,3)$ can be labeled by
the two half-integer spins $\quad S_1, S_2$ of $\quad SU(2) \times SU(2) \sim SO(1,3)$.

$$\text{scalar}: \quad S_1 = 0 \ , \ S_2 = 0$$

$$\text{Weyl spinor}: \quad S_1 = \frac{1}{2} \ , \ S_2 = 0 \qquad \Psi_\alpha \quad , \quad \alpha = 1, 2$$

$$\text{adjoint spinor}: \quad S_1 = 0 \ , \ S_2 = \frac{1}{2} \qquad \bar{\Psi}_{\dot{\alpha}} \quad , \quad \dot{\alpha} = 1, 2$$

$$\text{vector}: \quad S_1 = \frac{1}{2} \ , \ S_2 = \frac{1}{2} \qquad X^\mu \sigma_{\mu\alpha\dot{\alpha}}$$

Supercharges:

$$Q^I_\alpha \quad \text{are} \left(\frac{1}{2},0\right) \text{spinors}, \; \alpha = 1,2, \; I = 1,..,\mathcal{N}$$

$$\left(Q^I_\alpha\right)^\dagger = \bar{Q}_{I\dot{\alpha}} \quad \text{their adjoints;} \quad [P_\mu \, , Q] = 0$$

$U(\mathcal{N})$ is the 'R-symmetry' group

$$Q^I_\alpha = U^I_J(\mathcal{N}) \; Q^J_\alpha$$

$$\left\{Q^I_\alpha \;\; , \;\; \bar{Q}_{J\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} \; P_\mu \; \delta^I_J$$

$$\{Q^I_\alpha \;\; , \;\; Q^J_\beta\} = 2\varepsilon_{\alpha\beta} \; Z^{IJ} \quad \text{commutes, central charge}$$

Different members of a susy multiplet may have different R-charges J_R

$$[J_R \; , \; Q] \neq 0$$

Representations of susy algebra

- massless states:
  In a frame with

$$P_\mu = (E, 0, 0, E) \quad E > 0.$$

$$\left\{ Q^I_\alpha \, , \, \bar{Q}_{\dot{\beta} J} \right\} = \delta^I_J \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left\{ Q^I_2 \, , \, \left(Q^J_2\right)^\dagger \right\} = 0 \qquad \Rightarrow || \, Q^J_2 |\Psi > \, ||^2 = 0, \text{ for all } |\Psi > .$$

We are left with only the $Q^I_{\alpha=1}$

$$\left\{ \frac{Q^I_1}{2\sqrt{E}} \, , \, \frac{\bar{Q}_{1J}}{2\sqrt{E}} \right\} = \delta^I_J$$

they form $\mathcal{N}$ fermionic oscillators

Let $|h>$ be 'highest helicity state', $\bar{Q}_{1J}|h>= 0$

Then states of all other helicity are

$$\underbrace{|h> \ , \ Q_1^{I_1}\,|h> \ , \ \left(Q_1^{I_1}\,Q_1^{I_2}\right)\,|h> \ldots (Q_1^1 Q_1^2 \ldots Q_1^{\mathcal{N}})\,|h>}_{2^{\mathcal{N}} states}$$

Theorem (Nahm): A quantum field theory without gravity cannot contain massless states with maximal helicity $h > 1$
(with gravity the limit on allowed helicity is $|h| \leq 2$).

To get from $|h>$ to $|-h>$ in steps of $\frac{1}{2}$ one needs $\mathcal{N} = 4h$ factors Q_1^I
hence, maximal supersymmetry : (Note: for each value of I there are 2 complex (= 4 real) supercharges)

– $\mathcal{N} = 4$ for gauge fields $(h = 1)$ ⇒(16 supercharges in D=4)

– $\mathcal{N} = 8$ for gravity $(h = 2)$ ⇒(32 supercharges in D=4) number of massless states with given helicity in various multiplets:

helicity	$\mathcal{N}=1$ gauge	$\mathcal{N}=1$ chiral	$\mathcal{N}=2$ gauge	$\mathcal{N}=2$ hyper	$\mathcal{N}=4$ gauge
1	1	0	1	0	1
1/2	1	1	2	(1+1)	4
0	0	(1+1)	(1+1)	(2+2)	6
-1/2	1	1	2	(1+1)	4
-1	1	0	1	0	1

Let's discuss in a little bit more detail how this arises, say for the example of the $\mathcal{N}=2$ hypermultiplet with R-symmetry SU(2): Remember that only the components Q_1^1, Q_1^2 and their adjoints $\bar{Q}_{\dot{1}}^1, \bar{Q}_{\dot{1}}^2$ are at our disposal. The highest helicity component with $h = 1/2$ for $\mathcal{N}=2$ has two left-handed spinor-fields ($\mathbf{I}=\mathbf{1},\mathbf{2}$) both only with $\alpha = 1$. They transform (with respect to the $\mathbf{I}=\mathbf{1},\mathbf{2}$ indices) as spinor of the SU(2) R-symmetry.

$$\psi^{1,\mathbf{1}}, \quad \psi^{1,\mathbf{2}}$$

We shall see that the lowest helicity component with $h = -1/2$ for $\mathcal{N} = 2$ is of course again a SU(2) spinor with two componts ($\mathbf{I} = \mathbf{1}, \mathbf{2}$) and both appear only with $\alpha = \dot{\alpha} = \dot{1}$.

$$\bar{\psi}^{\dot{1},\mathbf{1}} \quad \bar{\psi}^{\dot{1},\mathbf{2}}$$

What means $\dot{1}$? Dont panic: $\dot{1} = 1$. We merely keep track what was set equal 1.

Applying our two Q's to the $h = 1/2$ fields for the first time we generate and define the 4 scalar $h = 0$ fields

$$\phi^{\mathbf{11}} = Q_1^{\mathbf{1}}\psi^{1,\mathbf{1}}; \quad \phi^{\mathbf{21}} = Q_1^{\mathbf{2}}\psi^{1,\mathbf{1}}; \quad \phi^{\mathbf{12}} = Q_1^{\mathbf{1}}\psi^{1,\mathbf{2}}; \quad \phi^{\mathbf{22}} = Q_1^{\mathbf{2}}\psi^{1,\mathbf{2}};$$

They transform like the direct product of two SU(2) doublets of the R-symmetry, that is like a j=0 plus j=1 of SU(2). Applying them for the second time

we get the two fields of the lowest helicity $h = -1/2$ component

$$\bar{\psi}^{\dot{1},\mathbf{1}} \sim \left(\sigma^{(0)1\dot{1}} Q_1^{\mathbf{2}}\right) Q_1^{\mathbf{1}} \psi^{1,\mathbf{1}} \sim \left(\sigma^{(0)1\dot{1}} Q_1^{\mathbf{1}}\right) Q_1^{\mathbf{2}} \psi^{1,\mathbf{1}};$$

$$\bar{\psi}^{\dot{1},\mathbf{2}} \sim \left(\sigma^{(0)1\dot{1}} Q_1^{\mathbf{2}}\right) Q_1^{\mathbf{1}} \psi^{1,\mathbf{2}} \sim \left(\sigma^{(0)1\dot{1}} Q_1^{\mathbf{1}}\right) Q_1^{\mathbf{2}} \psi^{1,\mathbf{2}}$$

Two things should become clear now

- The lowest helicity components so obtained are annihilated by $Q_1^{\mathbf{1}}$ and $Q_1^{\mathbf{2}}$.
- All steps can be reverted by applying the $\bar{Q}_{\dot{1}}^{\mathbf{1},\mathbf{2}}$ due to the fermi-oscillator anti-commutator of our $Q_1^{\mathbf{1},\mathbf{2}}$ and $\bar{Q}_{\dot{1}}^{\mathbf{1},\mathbf{2}}$.
  Therefore, the original states we recover from this procedure (or at least what remains of them after the state-projection implied by this procedure) is automatically annihilated by $\bar{Q}_{\dot{1}}^{\mathbf{1},\mathbf{2}}$

- massive states

rest frame $P^\mu = (M, 0, 0, 0)$

$$\left\{ Q^I_\alpha\ ,\ \bar{Q}_{\dot\beta J} \right\} = 2M\ \sigma^0_{\alpha\dot\beta}\ \delta^I_J \qquad \{ Q^I_\alpha\ ,\ Q^J_\beta \} = 2\varepsilon_{\alpha\beta}\ Z^{IJ}$$

block-diagonalize Z^{IJ}

$$Z^{IJ} = \begin{pmatrix} \begin{smallmatrix} 0 & Z_1 \\ -Z_1 & 0 \end{smallmatrix} & & \\ & \begin{smallmatrix} 0 & Z_2 \\ -Z_2 & 0 \end{smallmatrix} & \\ & & \ddots \end{pmatrix} = Z^{\bar I \hat I, \bar J \hat J} \qquad \begin{matrix} (\hat I, \hat J) = (1, 2) \\ (\bar I, \bar J) = (1, \ldots, \mathcal{N}/2) \end{matrix}$$

Define: $$Q^{\bar{I}}_{\alpha\,\pm} = \frac{1}{2}\left(Q^{\bar{I},1}_{\alpha} \pm Q^{\dagger\bar{I},2}_{\alpha}\right)$$

then $$\left\{Q^{\bar{I}}_{\alpha\pm}\,,\;\bar{Q}_{\dot{\beta}\bar{J}\pm}\right\} = \delta^{\bar{I}}_{\bar{J}}\,\sigma^0_{\alpha\dot{\beta}}\;\underbrace{(M \pm Z_{\bar{I}})}_{\geq 0}$$

Bogomolny bound $$M \geq |Z_{\bar{I}}|$$

If the bound is saturated, then

- $Q^{\bar{I}}_{\alpha+}$ or $Q^{\bar{I}}_{\alpha-}$ annihilate the state,
- BPS states are in small multiplets.
- The BPS-bound is a topological property, which cannot change if a continuous parameter, like a coupling constant, is changed
  The mass $M = |Z_I|$ is fixed by the central charge, a number but also an operator, determined by the susy-algebra.

Number of states with a given helicity in massive multiplets

helicity	$\mathcal{N}=1$ gauge	$\mathcal{N}=1$ chiral	$\mathcal{N}=2$ gauge	$\mathcal{N}=2$ gauge BPS	$\mathcal{N}=2$ hyper BPS	$\mathcal{N}=4$ gauge BPS
1	1	0	1	1	0	1
1/2	2	1	4	2	(1+1)	4
0	(1+1)	2	6	(1+1)	(2+2)	6
-1/2	2	1	4	2	(1+1)	4
-1	1	0	1	1	0	1

CPT conjugates are appearing with opposite helicity massive gauge-multiplets contain the same number of states as massless chiral and gauge multiplets together, i.e.

8 states = 4+4 from gauge + chiral for $\mathcal{N} = 1$,

16 states = 8+8 from gauge + hypermultiplet for $\mathcal{N} = 2$,

16 states = 16 from massless gauge-field alone for $\mathcal{N} = 4$

a sign that the Higgs mechanism of spontaneous gauge-symmetry breaking(gauge-field gets its mass and longitudinal dof by ’eating the Higgs’) also works in Susy

Spectrum for $\mathcal{N}=4$ $Susy$ with any gauge group, say $SU(N)$

1 gauge field , $h=1$, R-singlet A_μ, $\mu=0\ldots3$ $(=2$ d.o.f.$)$

4 Weyl spinors, R-fundamental λ^I_α, $I=1\ldots\mathcal{N}(=4)$, $\alpha=1..2$

6 scalars, SO(6) of $SU(\mathcal{N}=4)$ X^i $i=1\ldots\frac{\mathcal{N}(\mathcal{N}-1)}{2}(=6)$

They are all in the adjoint representation of gauge-group, here $SU(N)$;
R-symmetry is $SU(\mathcal{N}=4)$

leaves A_μ invariant,

acts on λ^I_α in the fundamental rep. of $SU(4)$ (= spinor rep. of $SO(6)$), i.e. by unitary 4x4 matrices, and

on the scalars by $SO(6)$ matrices. (note that $SU(4)\sim SO(6)$)

$\mathcal{N} = 4$ SYM

Fields $\quad A_\mu \quad , \quad \lambda^{I=1\ldots4} \quad , \quad X^{i=1\ldots6}$

all in the adjoint of the gauge group, say SU(N), e.g.

$$A_\mu = A_\mu^a\, T^a, \quad \text{with} \quad \left[T^a,\ T^b\right] = i f^{abc}\, T^c$$

$$\lambda^I = \lambda^{Ia}\, T^a. \qquad X^i = X^{ia}\, T^a$$

adjoint representation is D_A-dimensional, real, (D_A = number of group-generators)

Lagrangian

$$L = \frac{1}{g_{YM}^2}\, tr\Big[F^2 + (DX^i)^2 + i\,\bar\lambda\,\not{D}\,\lambda$$

$$- \sum_{i<j}[X^i,\ X^j]^2 - \lambda[X,\lambda] + \bar\lambda\,[X,\bar\lambda]\Big] + \frac{i\vartheta}{8\pi^2}\, tr\,(F\wedge F)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]$$
$$= \frac{i}{g}\,[D_\mu, D_\nu]$$
$$D_\mu = \partial_\mu - igA_\mu$$

Susy transformation, schematically

$$[Q, X] = \lambda,$$
$$\{Q, \lambda\} = F^+ + [X, X] \qquad \left(F^+ \equiv \sigma^{\mu\nu}\, F_{\mu\nu}\right)$$
$$\{Q, \bar{\lambda}\} = DX$$
$$[Q, A] = \lambda$$

Chiral field multiplets

consider Φ satisfying $[\bar{Q}, \Phi] = 0 \quad , \quad [Q, \Phi] \neq 0$

Φ is BPS, (because half of the supercharges annihilate it)

Φ is called chiral

Coulomb branch of vacua.

Susy vacuum has always energy $= 0$, otherwise susy is broken.

Fermions vanish,

kinetic terms vanish, field-strengths vanish

hence potential must also vanish for energy 0.

$$\sum_{i<j} tr \left[X^i, X^j\right]^2 = 0$$

Space of Susy vacua

$$\mathcal{M} = \left\{X | \left[X^i, X^j\right] = 0 \ \forall \ i, j\right\} / \text{(gauge symmetry e.g. } U(N))$$

Fix the gauge symmetry by taking all the X^i diagonal,

$$X^i = diag(X^i_1, \ldots X^i_N)$$

then the space of Susy vacua ist $6N$-dimensional.

If all the X^i_m, X^i_n are mutually different
then the ground state breaks the $U(N)$ gauge-symmetry
to $U(1)^N$ subgroup,

i.e. to N copies of electromagnetism: Coulomb-branch

The VEVs of the scalar fields $X^i_m - X^i_n$ with mass-dimension 1 define new mass-scales, which correspond to the masses of the gauge bosons of the broken gauge symmetries, analogous to the W-bosons in electroweak theory

$$m^{mn} = |X_m - X_n|$$

3-brane interpretation of $\mathcal{N}=4 \quad SYM$ in type II-B string theory in D=10

$\mathcal{N}=4 \quad SYM$ with $U(N)$ gauge-symmetry

is the world-volume theory

of N coincident 3-branes along directions $(1,2,3)$, say

$\rightarrow$ separate these into N parallel 3-branes Higgsing in string-language

In D=10 this can be done in 6 orthogonal transverse directions

$$\Delta y^i_{ab} = |X^i_a - X^i_b| \quad i = 4 \dots 9 \, (a = 1 \dots N))$$

The mass-matrix

$$m^i_{ab} = \frac{1}{\alpha'} \Delta y^i_{ab} \quad = (\text{string-tension}) \times \text{length}$$

comes from the strings stretched between the separated branes.

Montonen-Olive duality :

Define

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\vartheta}{2\pi}$$

The $\mathcal{N} = 4\ SYM$ theory with τ and gauge-group G is dual to

the $\mathcal{N} = 4\ SYM$ theory with $\tau' = -\frac{1}{\tau}$ and gauge-group ${}^L G$

(${}^L G$=Langlands-dual gauge-group of G)

This is a special case (for $a = 0, b = 1, c = -1, d = 0$) of the even larger

S-duality, where

$$\tau' = \frac{a\tau + b}{c\tau + d} \ , \quad \begin{array}{c} ad - bc = 1 \\ a \ldots d \text{ integer} \end{array}$$

Finally: **Conformal symmetry doubles supersymmetry**

Q^I_α, and K generate supersymmetry , and special conformal transformations, respectively

$$[Q^I_\alpha, K] = S^I_\alpha \neq 0, \text{ additional fermionic symmetry generators}$$

In the $\mathcal{N} = 4$ SYM theory, with its superconformal symmetry, there are then a total of 32 supersymmetries