AdS/CFT Correspondence
with applications to condensed matter

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1. Quantum Field Theory and Gauge Theory
2. Conformal Field Theory
3. Brief Introduction to Supersymmetry
4. General Relativity

SFB/TR 12: Symmetries and Universality in Mesoscopic Systems

III. General Relativity

SRT: constancy of velocity of light

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \text{invariant}$$

Minkowski space-time symmetry SO(1,3)
Euclidean space-time symmetry SO(4)

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \quad \Lambda^\mu_\nu \in SO(1,3) \quad \text{or} \in SO(4)$$

GRT: equivalence principle

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu, \quad g_{\mu\nu}(x) \text{ symmetric matrix}$$

(pseudo)-Riemannian geometry, curved space(-time)
example: 2-d sphere
embedded in 3-d flat space:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

but on 2-d sphere

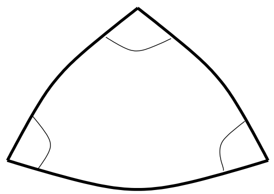
$$x_1^2 + x_2^2 + x_3^2 = R^2, \quad 2x_1 dx_1 + 2x_2 dx_2 + 2x_3 dx_3 = 0$$

eliminate x_3

$$dx_3 = -\frac{x_1 dx_1 + x_2 dx_2}{\sqrt{R^2 - x_1^2 - x_2^2}}$$

$$\begin{aligned} ds^2 &= dx_1^2 (1 + \frac{x_1^2}{R^2 - x_1^2 - x_2^2}) + dx_2^2 (1 + \frac{x_2^2}{R^2 - x_1^2 - x_2^2}) + 2 dx_1 dx_2 \frac{x_2 x_2}{R^2 - x_1^2 - x_2^2} \\ &= R^2 (\sin^2 \theta d\varphi^2 + d\theta^2) \end{aligned}$$

the 2-sphere has positive curvature
$\sum_{i=1}^{3} \varphi_i > \pi$ in triangles



Lobachewsky (1826) space is the analogon with negative curvature.

Example: hyperboloid (2-pseudosphere) Beltrami, 1868 in 3-d flat Minkowski-space:

$$ds^2 = dx_1^2 + dx_2^2 - dx_3^2$$

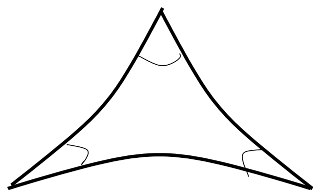
$$x_1^2 + x_2^2 - x_3^2 = -R^2, \quad 2x_1dx_1 + 2x_2dx_2 - 2x_3dx_3 = 0$$

eliminate x_3

$$dx_3 = \frac{x_1dx_1 + x_2dx_2}{\sqrt{R^2 + x_1^2 + x_2^2}}$$

$$ds^2 = dx_1^2(1 - \frac{x_1^2}{R^2 + x_1^2 + x_2^2}) + dx_2^2(1 - \frac{x_2^2}{R^2 + x_1^2 + x_2^2}) - 2dx_1dx_2\frac{x_2x_2}{R^2 + x_1^2 + x_2^2}$$
$$= R^2(\sinh^2\theta d\varphi^2 + d\theta^2)$$

the 2-pseudosphere has in triangles $\sum_{i=1}^{3}\varphi_i < \pi$



Einstein's theory

- physics is independent of the coordinates used
  -principle of general covariance,
  -physics expressible in tensor-equations

- gravity is locally indistinguishable from accelerated frames of reference, i.e. local duality of gravitational forces with D'Alembertian inertial forces (equivalence principle)

Local tensor eqs. of SRT promoted to tensor eqs. of GRT by replacing tensors in Minkowski space by tensors in pseudo-Riemannian space, e.g. $\eta_{\mu\nu} \to g_{\mu\nu}(x)$ used for raising and lowering indices.

What's the replacement for the covariant vector $\partial_\mu = \frac{\partial}{\partial x^\mu}$?
It's the covariant derivative D_μ
Depends on the object on which it acts.

For scalar ϕ, $\partial_\mu \phi(x)$ is a vector hence, simply $D_\mu \phi = \partial_\mu \phi$ for scalar

and also e.g. $D_\mu \left(V^\nu V_\nu\right) = \partial_\mu \left(V^\nu V_\nu\right)$
and

$$\begin{aligned}
\partial_\nu V^\mu &\quad \to \quad D_\nu V^\mu \equiv \partial_\nu V^\mu + \Gamma^\mu_{\nu\kappa} V^\kappa \\
\partial_\nu V_\mu &\quad \to \quad D_\nu V_\mu \equiv \partial_\nu V_\mu - \Gamma^\kappa_{\nu\mu} V_\kappa \\
\partial_\mu \eta_{\kappa\lambda} = 0 &\quad \to \quad D_\mu g_{\kappa\lambda}(x) = 0 \\
\partial_\mu \eta^{\kappa\lambda} = 0 &\quad \to \quad D_\mu g^{\kappa\lambda}(x) = 0
\end{aligned}$$

Explicitely

$$\begin{aligned}
\partial_\mu g^{\kappa\lambda} + \Gamma^\kappa_{\mu\alpha} g^{\alpha\lambda} + \Gamma^\lambda_{\mu\alpha} g^{\kappa\alpha} = 0 \\
\partial_\mu g_{\kappa\lambda} - \Gamma^\alpha_{\mu\kappa} g_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda} g_{\kappa\alpha} = 0
\end{aligned}$$

Can be solved for 'affine connection' $\Gamma^{\cdot}_{\cdot\cdot}$ with result

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_{\rho} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\rho} - \partial_{\sigma} g_{\nu\rho}) + \Gamma^{\mu}_{[\nu\rho]}$$

$\Gamma^{\mu}_{\{\nu\rho\}}$ (Levi-Civita connection, its components the Christoffel-symbols)

- not a tensor, but ${\Gamma^{\mu}}_{[\nu\rho]}$ is (named torsion-tensor, assumed to vanish in GRT), and more generally the difference between any two connections is a tensor;
- can be transformed to 0 at any picked, isolated point, but not also in its neighbourhood
- plays the role of a gauge-potential (and can be given 1-form properties) for gravity
- the field-strength (which can be given 2-form properties, see later in this lecture) then is the Riemann curvature-tensor

$$(R^{\mu}_{\nu})_{\rho\sigma} = \partial_\rho(\Gamma^{\mu}_{\nu})_\sigma - \partial_\sigma(\Gamma^{\mu}_{\nu})_\rho + (\Gamma^{\mu}_{\lambda})_\rho(\Gamma^{\lambda}_{\nu})_\sigma - (\Gamma^{\mu}_{\lambda})_\sigma(\Gamma^{\lambda}_{\nu})_\rho$$

contractions of the Riemann tensor:

- Ricci-tensor $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$
- Ricci-scalar $R = R^{\mu}{}_{\mu}$

The much easier Cartan method for computing curvature

see e.g. Misner, Thorne and Wheeler, 'Gravitation' sections 14.5, 14.6

1. Introduce D linearly independent basis-1-forms $e^{\hat{\mu}}(x)$ such that metric can be written

$$ds^2 = e^{\hat{\mu}} \otimes e^{\hat{\nu}} \eta_{\mu\nu}$$

2. basis 1-forms $e^{\hat{\mu}}(x)$ are covariantly const. , like the metric which they reexpress. They now permit to define the promissed connection 1-forms whose components are the connection coefficients via

$$\omega^{\mu}{}_{\lambda} = \Gamma^{\mu}{}_{\lambda\kappa} e^{\hat{\kappa}}$$

These are also called spin-connections and appear as connection coefficients and 1-forms in the relation for the covariantly constant $e^{\hat{\mu}}$

$$d\, e^{\hat{\mu}} + \omega^{\mu}{}_{\lambda} \wedge\ e^{\hat{\lambda}} = 0 \qquad \mathrm{spin-connection}$$

This fixes the $\omega_{\mu\nu}$ uniquely, we can often determine them by guessing. In difficult cases calculate the 2-form $de^{\hat{\alpha}}$ and expand it out in the basis 1-forms as $-c^{\alpha}_{|\mu\nu|}e^{\hat{\mu}} \wedge e^{\hat{\nu}}$ and then apply

$$\omega_{\mu\nu} = \frac{1}{2}(c_{\mu\nu\alpha} + c_{\mu\alpha\nu} - c_{\nu\alpha\mu})e^{\hat{\alpha}}$$

Moreover, from $d(ds^2) = 0$ and $d\eta_{\mu\nu} = 0$ follows $\omega_{\mu\nu} = -\omega_{\nu\mu}$

3. curvature 2-form

$$\mathcal{R}^{\mu\nu} = d\,\omega^{\mu\nu} + \omega^{\mu}{}_{\lambda} \wedge\ \omega^{\lambda\nu}$$
$$= R^{\mu\nu}{}_{|\rho\sigma|}\, e^{\hat{\rho}} \wedge e^{\hat{\sigma}} \quad \mathrm{Riemann-Curvaturetensor}$$
$$R^{\mu}{}_{\nu} = R^{\mu\rho}{}_{\nu\rho} \qquad \mathrm{Ricci-Tensor}$$
$$R = R^{\mu}{}_{\mu} \quad .$$

Example AdS_5: In Poincare coordinates: $ds^2 = \frac{L^2}{z^2}\left((dz)^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right)$

$$ds^2 = \left(e^{\hat{z}}\right)^2 + e^{\hat{\mu}} \otimes e^{\hat{\nu}}\ \eta_{\mu\nu} \quad ; \quad e^{\hat{z}} = L\frac{dz}{z} \quad , \quad e^{\hat{\mu}} = L\frac{dx^\mu}{z}$$

- Compatibility of basis 1-forms
  Notation: distinguish the $D-1$ values of the Poincare-indices $\hat{\mu},\ \hat{\nu}$ and extra index $\hat{z}$;
  $\hat{a},\ \hat{b}$ denote both $\hat{\mu}$ and $\hat{z}$ and correspond to the $\mu\ \nu$ in the **general**

formulae outside the present example.

$$de^{\hat{a}} + \omega^{\hat{a}}{}_{\hat{b}} \wedge e^{\hat{b}} = 0$$

1. $de^{\hat{z}} = 0 \quad \Rightarrow$ either $\omega^{\hat{z}}{}_{\hat{k}} = 0$ or $\omega^{\hat{z}}{}_{\hat{\mu}} \sim e^{\hat{\mu}}$ because $e^{\hat{\mu}} \wedge e^{\hat{\mu}} = 0$

2. $de^{\hat{\mu}} + \frac{L}{z^2}\, dz \wedge dx^{\mu} = 0$ rewritten $de^{\hat{\mu}} + \frac{1}{L}\, e^{\hat{z}} \wedge e^{\hat{\mu}} = 0$

   try $\omega^{\hat{\mu}\hat{z}} = -\frac{1}{L}\, e^{\hat{\mu}} = -\omega^{\hat{z}\hat{\mu}}$ yes, that works !

   and then $\omega^{\hat{\mu}\hat{\nu}} = 0$

- On to the curvature 2-form

$$\mathcal{R}^a{}_b = d\,\omega^a{}_b + \omega^{ac} \wedge \omega_{cb} = R^a{}_{b|cd|}\, e^{\hat{c}} \wedge e^{\hat{d}}$$

$$\mathcal{R}^{\hat{\mu}\hat{\nu}} = 0 \;+\; \omega^{\hat{\mu} z} \wedge \omega_z{}^{\hat{\nu}} = -\frac{1}{L^2}\, e^{\hat{\mu}} \wedge e^{\hat{\nu}}$$

$$\mathcal{R}^{\hat{\mu}\hat{z}} = d\omega^{\hat{\mu}\hat{z}} = -\frac{1}{L}\, de^{\hat{\mu}} = +\frac{1}{L^2}\, e^{\hat{z}} \wedge e^{\hat{\mu}}$$

$$R^{\mu\nu}{}_{ab} = -\frac{1}{L^2}\, \left(\delta^\mu{}_a\, \delta^\nu{}_b - \delta^\mu{}_b\, \delta^\nu{}_a\right)$$

$$R^{\hat{\mu}\hat{z}}{}_{ab} = -\frac{1}{L^2}\, \left(\delta^\mu{}_a\, \delta^r{}_b - \delta^\mu{}_b\, \delta^z{}_a\right)$$

$$R^{ab}{}_{cd} = -\frac{1}{L^2}\, \left(\delta^a{}_c\, \delta^b{}_d - \delta^a{}_d\, \delta^b{}_c\right)$$

hence, with $D = 5$ for AdS_5

$$R^a{}_b = -\frac{1}{L^2}(D-1)\delta^a_b$$
$$R = -\frac{D(D-1)}{L^2}$$

From

$$R_{ab} - \frac{1}{2}\, g_{ab}\, R = -\Lambda g_{ab}$$
$$R - \frac{D}{2}\, R = -\Lambda\ D \quad , \quad R = \frac{2D}{D-2}\,\Lambda$$

we find for the AdS metric

$$R = -\frac{D(D-1)}{L^2} \quad , \quad \Lambda = -\frac{D-2}{L^2}\,\frac{D-1}{2} = -\frac{6}{L^2} \ \text{for} \ \mathrm{AdS}_5$$

Action principle (in D space-time dimensions) (Einstein, Hilbert)

$$S_{grav} = \frac{1}{16\pi G} \int d^D x \sqrt{-g} (R - 2\Lambda)$$

From

$$\frac{\delta (S_{grav} + S_{matter})}{\delta g^{\mu\nu}} = 0$$

and definition of energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$$

follows Einstein's equation of GRT

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}$$

with the Einstein-tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Penrose diagram to discuss space-times

- Make a conformal transformation which brings infinity to a finite distance, and preserves the causal structure (i.e. light travels on 45 degrees lines)

- D=2 Minkowski:

$$ds^2 = -dt^2 + dx^2$$

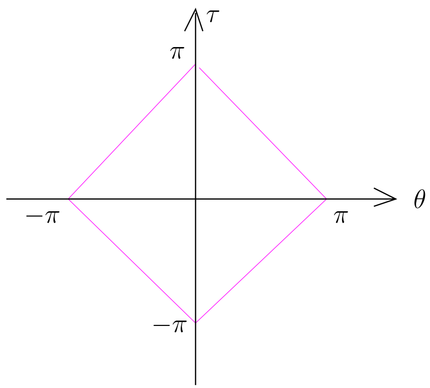
$$u_\pm = t \pm x$$

$$ds^2 = -du_+ du_-$$

$$u_\pm = \tan \tilde{u}_\pm, \quad \tilde{u}_\pm = \frac{\tau \pm \theta}{2}$$

$$\to ds^2 = \frac{-d\tau^2 + d\theta^2}{4\cos^2 \tilde{u}_+ \cos^2 \tilde{u}_-}$$

$$|\tilde{u}_\pm| \leq \frac{\pi}{2} \quad \to \quad |\tau \pm \theta| \leq \pi$$



- D=4 Minkowski

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

$-dt^2 + dr^2$ is like in 2-d, except $x \to r > 0$, i.e. now $0 \leq \theta \leq \pi$

- D-dim, deSitter space-time embedded on Lorentzian version of sphere

$$-x_0^2 + \sum_{i=1}^{D-1} x_i^2 + x_D^2 = L^2$$

in flat (1,D) Minkowski space-time:

$$ds^2 = -dx_0^2 + \sum_{i=1}^{D} dx_i^2$$

- D-dim, Anti-deSitter (AdS$_D$) space-time with radius L embedded on Lorentzian version of a (D+1)-Lobachevsky space

$$-x_0^2 + \sum_{i=1}^{D-1} x_i^2 - x_D^2 = -L^2$$

with the non-contractible time-like cycle $x_0^2 + x_D^2 = L^2 + \sum_{i=1}^{D-1} x_i^2 > 0$ i.e. in flat (2,D-1)(generalized) Minkowski space-time:

$$ds^2 = -dx_0^2 + \sum_{i=1}^{D-1} dx_i^2 - dx_D^2$$

it's manifestly invariant under group $SO(2, D-1)$ = conformal group in D-1 dimensions **used in AdS/CFT correspondence.**

- AdS$_D$ in Poincaré-coordinates: $ds^2 = L^2 \frac{dz^2 + dx^\mu dx_\mu}{z^2}$

$$
\begin{aligned}
x_D &= \frac{z}{2}(1 + \frac{L^2 + \vec{x}^2 - t^2}{z^2}) \\
x_0 &= \frac{L}{z} t \\
x_i &= \frac{L}{z} \quad (\vec{x})_i, \qquad i = 1, 2, ..D - 2 \\
x_{D-1} &= \frac{z}{2}(1 - \frac{L^2 - \vec{x}^2 + t^2}{z^2})
\end{aligned}
$$

$-\infty < t, x < \infty$, $0 \leq z \leq \infty$ meaning that only half of AdS$_D$ is covered by these coordinates.

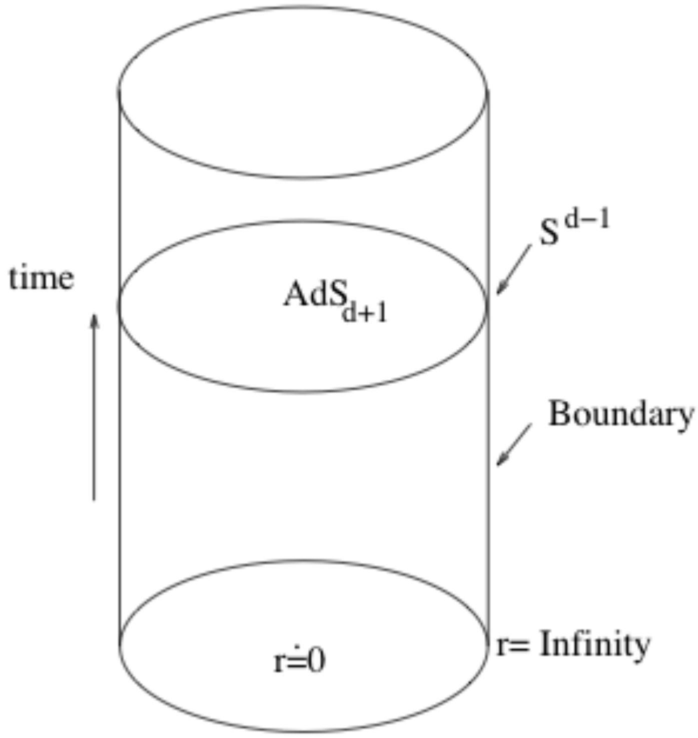
Often used instead of z: $r = \frac{1}{z}$

For z or r const.: it's the metric of (D-1)-dimensional Minkowski space-time with warp-factor (gravitational potential) $1/z^2 = r^2$.

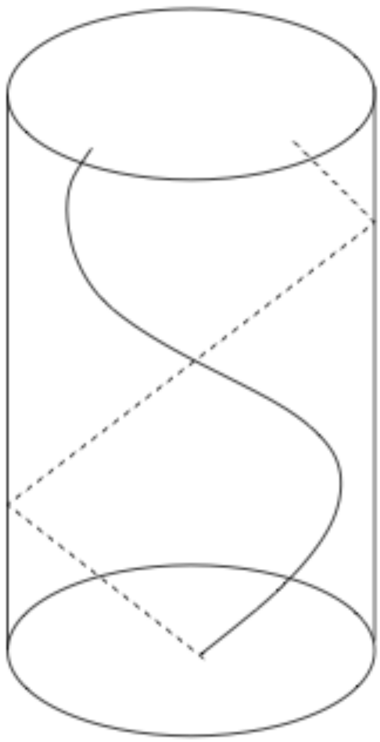
$z = 0$; or $r = \infty$: boundary of AdS$_D$ with singularity (but $z^2 ds^2$ remains regular and has boundary at $z = 0$);

$z = \infty$, or $r = 0$: horizon of AdS$_D$, but no singularity there; thus extension to negative r, the other half of AdS space-time, is possible

Penrose diagram of AdS_D space-time in coordinates (r, t, S^{D-2}):



(a)


(b)

- global coordinates

$$x_0 = L\cosh\rho\cos\tau$$

$$x_D = L\cosh\rho\sin\tau$$

$$x_i = L\hat{x}_i\sinh\rho, \quad i = 1..(D-1) \quad \sum_{i=1}^{D-1}\hat{x}_i^2 = 1$$

$$ds^2 = L^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{D-2}^2\right)$$

Beware: τ is periodic in $0 \leq \tau < 2\pi$: closed timelike curves!

Avoided by taking universal cover: τ = real line

D-dimensional sphere just for comparison

$$d\Omega_D^2 = L^2\left(\cos^2\rho dw^2 + d\rho^2 + \sin^2\rho d\Omega_{D-2}^2\right)$$

- with $\tan\theta = \sinh\rho$ where $0 \leq \theta < \frac{\pi}{2}$ (except $D = 2$ where $|\theta| \leq \frac{\pi}{2}$)

$$ds^2 = \frac{L^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2_{D-2})$$

- As we know already this solves Einstein's equation with a negative cosmological constant

$$\Lambda = -\frac{(D-2)(D-1)}{2L^2}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu}$$

Summary and some Remarks on the Action Functional of Gravity

- Gravity has a lot in common with gauge theories

  affine connection corresponds to gauge-potential,
  Riemann curvature tensor corresponds to gauge-field strength

- **But** the form of it's Lagrange-density $R = R^{\mu}{}_{\kappa\mu\lambda} g^{\kappa\lambda}$ has no analog in gauge theories

- $S = \int \sqrt{-g}(R - 2\Lambda) d^4x$ is not fundamental but merely a long-wavelength effective action; higher order terms with R^2, $R^{\mu\kappa} R_{\mu\kappa}$ etc. are surely there, but carry additional prefactors l_{Planck}^2 on dimensional grounds, and therefore remain negligible as long as $|R| << l_{Planck}^{-2}$. Still, this spells doom to all attempts to take Einstein's form as complete and to seek its direct quantization. It would be like adopting Navier-Stokes' equations as ultimate truth and attempting, by its means alone, to understand the microscopic theory of fluids.

- D-dimensional anti-deSitter space-time

1. needs a negative cosmological constant,

2. has constant negative curvature, and

3. has the same $SO(2, D-1)$ symmetry group as conformal field-theory in dimension $D-1$

Black holes and p-branes

Schwarzschild (1916)

- Solution of Einstein's equation in 4-D in vacuum, radially symmetric

$$R_{\mu\nu} = 0$$

$$ds^2 = -(1 - \frac{2MG}{r})dt^2 + \frac{dr^2}{1 - \frac{2MG}{r}} + r^2 d\Omega_2^2$$

- Far away: $ds^2 = -dt^2 + d\vec{x}^2$ hence t is time of far away inertial observers

- r is defined by area $4\pi r^2$ of sphere at r around the center

- Newtonian limit for nonrelativistic velocity, and weak gravitational field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} << 1$$

In this limit

$$ds^2 = -(1 + 2U_N(r))dt^2 + (1 - U_N(r))d\vec{x}^2$$

and $d\vec{x}^2 = dr^2 + r^2 d\Omega_2^2$ with

- Newtonian potential $U_N = -\frac{1}{2}(g_{00} + 1) = -\frac{GM}{r}$
- has a point-source at $r = 0$ (like Newtonian potential) where metric is singular
- as long as $T_{\mu\nu} = 0$ for $r > r_0$ and rotational symmetry is guaranteed the solution must be Schwarzschild for $r > r_0$ (Birckhoff's theorem)
- Horizon $g_{00} = 0$ at $r_H = r_S$ with Schwarzschild radius $r_S = 2GM$
- second singularity at $r = r_S$ is merely a coordinate singularity
  Ricci scalar at r_S: $R \sim \frac{1}{r_S^2}$

- light propagation $ds^2 = 0$ in radial direction

$$\frac{dr}{dt} = \pm(1 - \frac{2MG}{r})$$

near r_S: for $r \to +2MG$

$$dt \simeq \pm 2MG \frac{dr}{r - 2MG}$$
$$t \simeq -2MG \ln(r - 2MG) \to \infty$$

- viewed from far away, inwards travelling light takes infinitely long to reach the horizon; but people living inside the horizon see the light-signal appearing after a finite amount of <u>their</u> time and, more generally, dont see any trace of the horizon. The domain $r > r_S$ is cut off from communication with $r < r_S$ (instead, quantum mechanically there is Hawking radiation from the horizon for $r > r_S$, but not for $r < r_S$)

- In his own local time an infalling observer reaches the horizon after a finite time, and sees finite gravitational forces there.

- Thus, due to their usually different local times, different observers may see very different and sometimes apparently contradictory things

- Kruskal-Szekeres coordinates (1960) in 3 steps:

1. tortoise-coordinate r_*

$$\frac{dr}{1-\frac{2MG}{r}} = dr_*$$

$$r_* = r + 2MG\ln(\frac{r}{2MG} - 1)$$

$$ds^2 = (1-\frac{2MG}{r})(-dt^2 + dr_*^2) + r^2(r_*)d\Omega_2^2$$

2. null-coordinates

   $u = t - r_*, \qquad v = t + r_*$

   hence $ds^2 = 0$ implies $\qquad u = const$ or $v = const$

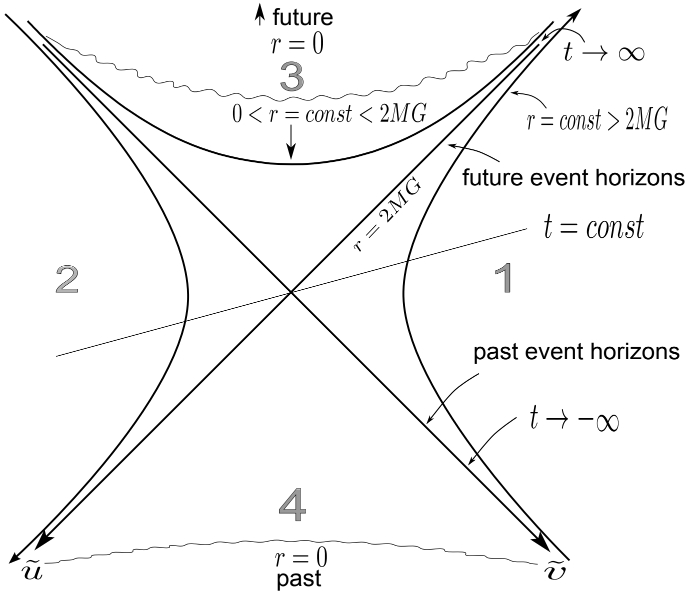
3. Kruskal coordinates

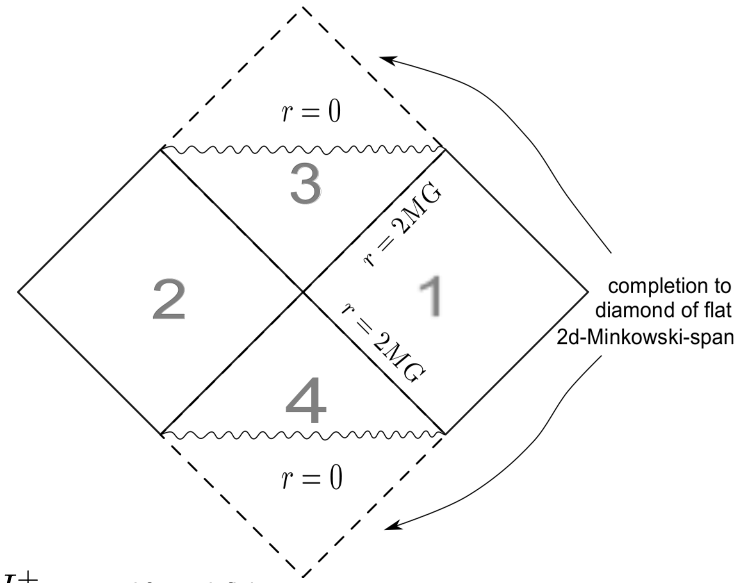
$$\tilde{u} = -4GM \exp \frac{-u}{4GM}$$
$$\tilde{v} = +4GM \exp \frac{-v}{4GM}$$

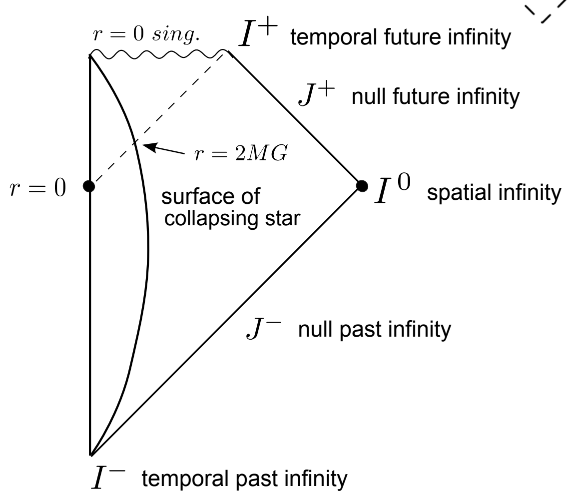
$$r > 2MG \quad \Rightarrow -\infty < r_* < +\infty \quad -\infty < u < +\infty \quad -\infty < v < +\infty$$
$$\Longrightarrow \quad -\infty < \tilde{u} < 0 \quad 0 \leq \tilde{v} < +\infty$$

$$ds^2 = -\frac{2MG}{r} \exp(-\frac{r}{2MG}) d\tilde{u} d\tilde{v} + r^2(\tilde{u}, \tilde{v}) d\Omega_2^2$$

- The Kruskal-solution can be analytically extended to the whole $\tilde{u}, \tilde{v}$ plane. It then gets a second domain with $r > 2MG$
- Penrose diagram for $d\tilde{u}d\tilde{v}$ part of metric looks like the diamond of flat Minkowski space cut off by the future and past singularity at $r = 0$


future
$r = 0$
3
$0 < r = const < 2MG$
$t \to \infty$
$r = const > 2MG$
future event horizons
$r = 2MG$
$t = const$
2
1
past event horizons
$t \to -\infty$
4
$\tilde{u}$
$r = 0$
past
$\tilde{v}$


$r = 0$
3
$r = 2MG$
2
1
$r = 2MG$
4
$r = 0$
completion to
diamond of flat
2d-Minkowski-span


$r = 0$ sing.
I^+ temporal future infinity
J^+ null future infinity
$r = 2MG$
$r = 0$
surface of
collapsing star
I^0 spatial infinity
J^- null past infinity
I^- temporal past infinity

Charged black holes

Reissner-Nordstrøm metric

$$ds^2 = -\Delta(r)dt^2 + \frac{dr^2}{\Delta(r)} + r^2 d\Omega_2^2$$

$$F_{rt} = \partial_r A_t - \partial_t A_r = \frac{Q}{r^2}$$

$$\Delta(r) = 1 - \frac{2MG}{r} + \frac{Q^2 G}{r^2}$$

Event horizon : $\Delta(r) = 0$:

1. If $M > Q/\sqrt{G}$:
   two horizons (outer and inner)

$$r = r_\pm = GM \pm \sqrt{(GM)^2 - GQ^2}$$

$$\Delta = (1 - \frac{r_+}{r})(1 - \frac{r_-}{r})$$

2. If $M < Q/\sqrt{G}$
   No horizon remains; just a naked singularity at $r = 0$.

   Electric charge plays only a minor role for physical black holes.
   But in quantum gravity charges of D-branes also produce Reissner Nordstrøm black holes and branes.

3. Special case $M = Q/\sqrt{G}$
   'Extremal' R-N black hole $\Delta = (1 - \frac{GM}{r})^2$

Change of radial coordinate near horizon: $r = GM + \tilde{r}$

$$ds^2 = -\frac{1}{(1+\frac{GM}{\tilde{r}})^2}dt^2 + (1+\frac{GM}{\tilde{r}})^2(d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2)$$
$$= -\frac{1}{H^2(\tilde{r})}dt^2 + H^2(\tilde{r})(d\tilde{r}^2 + \tilde{r}^2 d\Omega_2)^2$$

$H(\tilde{r}) =$ harmonic function in $d = 3$ dimensions . $\Delta H = -4\pi GM\delta^{(3)}(r)$

Reissner-Nordstrøm black hole in Anti deSitter space-time

AdS_D metric can be written as

$$ds^2 = -(1 - \frac{2\Lambda r^2}{(D-1)(D-2)})dt^2 + \frac{dr^2}{1 - \frac{2\Lambda r^2}{(D-1)(D-2)}} + r^2 d\Omega_3^2$$

deforms R-N metric to $\quad ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega_3^2 \quad$ with

$$\Delta = 1 - \frac{2gM}{r^{D-3}} + \frac{Q^2 g}{r^{D-2}} - \frac{2\Lambda r^2}{(D-1)(D-2)}$$

where we wrote Newton's constant in D dimensions, and also r and M and Q in units of the AdS-radius as

$$g = \frac{G_D^{Newton}}{L^{D-2}}; \quad \frac{r}{L} \to r; \quad ML \to M; \quad \frac{Q}{L^{D/2-2}} \to Q;$$

The only other conserved quantity one may add is angular momentum J: Kerr black hole. But we shall not discuss that here.

Black branes in D=4 dimensions there are only black

- 0-branes (i.e. black holes)
- 1-branes (i.e. black strings)
- 2-branes (i.e. black domain walls)

In d spatial dimensions p-branes are always characterized by harmonic functions $H(r) \sim 1/r^{D-3-p}$ with spherical symmetry in the coordinates transverse to the branes; e.g. in our $d = 3$

$$\begin{array}{lll} black\ hole & p = 0 & H \sim 1/\sqrt{x^2 + y^2 + z^2} \\ black\ string & p = 1 & H \sim \ln\sqrt{x^2 + y^2} \\ black\ membrane & p = 2 & H \sim |x| \end{array}$$

More possibilities of black p-branes exist for $D > 4$

e.g. 3-brane whose 4-D current couples to a 4-form potential with a 5-form gauge field $F_5 = dA_4$ and we take D=10 (since its relevant for string-theory):

$$ds^2 = H(r)^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^{\frac{1}{2}} \sum_{i=1}^{6} dy_i^2$$

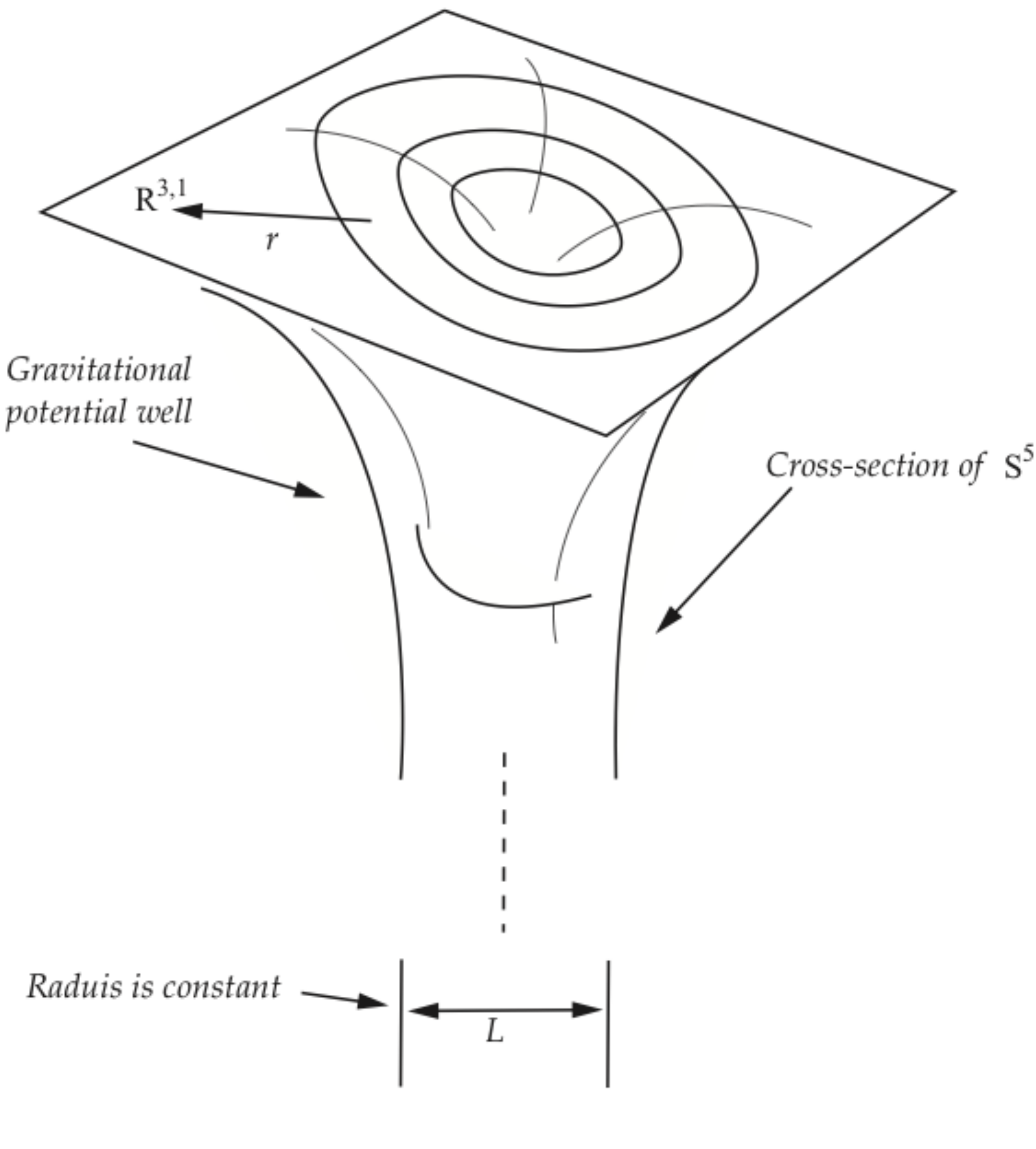
$$\sum_{i=1}^{6} dy_i^2 = dr^2 + r^2 d\Omega_5 \quad H(r) = 1 + \frac{L^4}{r^4}$$

$$A_{0123} \sim (H^{-1} - 1)$$

Far away from the brane $H(r) \to 1$ the solution becomes $R^{1,9}$. Near the horizon (which is at r=0) for $r << L$ the metric becomes $AdS_5 \times S_5$

$$ds^2 = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + L^2 \frac{dr^2}{r^2} + L^2 d\Omega_5$$

We have a throat-like geometry depicted in a picture I borrow from McGreevy (loc.cit.).



$R^{3,1}$
r
Gravitational potential well
Cross-section of S^5
Raduis is constant
L

Thus: AdS geometry appears if we go close close to horizons.

We can also put black holes inside AdS space-time, multiplying dt^2 by it's emblackening factor f:

$$ds^2 = \frac{-f dt^2 + d\vec{x}^2}{\sqrt{H(r)}} + \sqrt{H(r)} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2 \right)$$

$$H(r) = 1 + (\frac{L}{r})^4$$

$$f(r) = 1 - (\frac{r_H}{r})^4$$