I. Quantum Field Theory and Gauge Theory

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VI. A hand-waving derivation of AdS/CFT

A hand-waving derivation of AdS/CFT

- Recall that Dp-branes are subspaces with $(p+1) - \mathrm{dim}$ worldvolumes, where open strings can end

- light open strings are short, only weakly excited, and are localized on and close to the branes, whose fluctuations they describe

- Dp branes carry their own charges (RR-charges, potentials C_{p+1})) satisfying Dirac quantization (like electric charges in the presence of magnetic monopoles).

$$dC_{p+1} = F_{p+2} = *F_{8-p} = *dC_{7-p};$$

$p = 3 : F_5 = {}^*F_5$ self dual, $\quad \int_{S_5} F_5 = \int_{S_5} {}^*F_5 = N$

Some memory refreshing of differential forms:

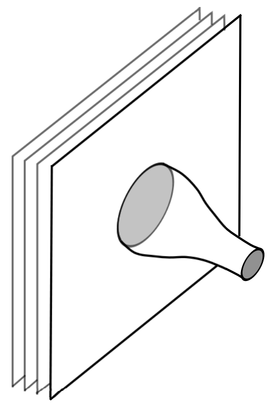
In a D=4 world-volume magnetic 2-flux $F^{(2)}$ is dual to
electric 2-flux $*F^{(2)}$ (differ by exchange of electric and magnetic)
$\int_{S_2} F_2 = 0$ (no magnetic monopoles enclosed by S_2)
$\int_{S_2} {}^*F_2 = Q$ ($Q =$ charge enclosed by S_2)

In IIB string theory in a D=10 world-volume the RR-field $F^{(5)}$ is self-dual

$$({}^*F^{(5)})_{\mu\nu\kappa\lambda\rho} = \varepsilon_{\mu\nu\kappa\lambda\rho}{}^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}} \; F^{(5)}_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}} = F^{(5)}_{\mu\nu\kappa\lambda\rho}$$

- On the world-volume of N coincident p-branes, open strings are free to begin and end on anyone of those;
  described by $N \times N$ unitary matrices, forming a $U(N)$-group:
  a single $U(1) = U(N)/SU(N)$-factor describes collective motion,
  the remaining internal symmetry $SU(N)$ – a Yang Mills theory;
  if all the N branes are split apart, open strings get massive (Higgs mechanism) and $U(1) \times SU(N)$ is broken down to $U(1)^N$

- emission of RR-bosons and of gravitons



$N \; p - branes$

tree-diagram of N Dp branes, emitting a closed string (graviton)
or, via an equivalent 1-loop-diagram,
an open string (RR gauge boson)

disk-amplitude (with number of handles h=0, and boundaries b=1)
$\sim$ tension of brane $\sim \alpha'^{-D/2} N g_S^{-2+2h+b} \sim \alpha'^{-D/2} N g_S^{-1}$

- back-reaction of N coincident Dp-branes on space-time: from Einstein's equations in $D = 10$

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 8\pi \underbrace{G_N}_{\sim g_S^2\ \alpha'^4} \overbrace{T_{\mu\nu}}^{\sim \frac{1}{g_S}\ N\ \alpha'^{-5}} \sim \alpha'^{-1} \underbrace{g_S N}_{\equiv \lambda/4\pi}$$

  - small 't Hooft coupling λ: back-reaction negligible, black brane metamorphs to $\rightarrow$ Dp brane

    then we have very weakly interacting gravitons (closed strings) near to and far from the brane
    and interacting gauge-bosons (weakly excited open strings) but only on the brane

  - large 't Hooft coupling λ: now the back-reaction strong:

    the N Dp-branes now form a black brane, (take $r = 0$ on horizon) , collapsed transverse to and extended along original Dp branes.

Black brane metric for special case p=3

3+1 dim. world volume $x^\mu, \quad \mu = 0 \dots 3$

transverse coordinates y_i, horizon at $y_i = 0, i = 1 \dots 6$,

i.e. at $r = 0$, $\quad r^2 = \sum_{i=1}^{6} y_i^2$

Form of metric

$$ds^2 = \frac{\eta_{\mu\nu}}{\sqrt{H(r)}} dx^\mu \, dx^\nu + \sqrt{H(r)} \sum_{i=1}^{6} dy_i^2$$

$H(r)$ is a harmonic function of the transverse y-coordinates

$$\sum_{i=1}^{6} dy_i^2 = dr^2 + r^2 d\Omega_5^2 \qquad \text{in polar coordinates}$$

$$H(r) = 1 + \frac{L^4}{r^4} \begin{cases} 1 & \text{for } r \to \infty \\ \sim \frac{L^4}{r^4} & \text{for } r \to 0 \\ & \\ L^4 = 4\pi g_s N \alpha'^2 & \end{cases}$$

In addition there is the self-dual RR 5-form field $F^{(5)}(r)$

in the 6-dimensional space transverse to the brane.

Its flux counts the number of 3-branes sourcing it.

$$\int_{S_5} F^{(5)}(r) = N$$

Far from the brane, for $r \to \infty$, there is the boundary of $AdS_5 \times S_5$ at $r \sim L$.
It is reached by light in a finite time and therefore an excellent place from which to observe what happens in AdS_5.
Beyond $r \sim L$ for $r \gg L$ the metric becomes $D = 5$ Minkowski.
Close to the brane it becomes $AdS_5 \times S_5$

$$r \ll L \qquad ds^2 = \underbrace{\frac{r^2}{L^2}\,\eta_{\mu\nu}\,dx^\mu x^\nu + L^2\,\frac{dr^2}{r^2}}_{AdS_5} + \underbrace{L^2\,d\Omega_5^2}_{S_5}$$

where the black brane itself hides behind the horizon, which is at r=0.
(The boundary of the AdS, as purified in the present limit, would now be at $r = \infty$.)

Due to infinite redshift at the horizon, even arbitrarily high-energy and short-wavelength excitations (which really means all of type IIB theory) become visible looking from the boundary on a sufficiently close neighbourhood of the horizon.

large 't Hooft coupling λ
(large back-reaction, big black-brane formation)

small λ
(small back-reaction,
N D_3-branes)

Low-energy excitations

near brane :
All excitations of string get
red-shifted to low energy for $r \to 0$;
quantum-gravity (complete II B superstring-theory)
on $AdS_5 \times S_5$ background.

open strings, $\mathcal{N} = 4$
SU(N) Super-Yang-Mills
gauge theory on D=4
$r \to \infty$ boundary of AdS_5

far from brane: except for gravity
ultralow energy excitations cannot
escape from brane to $r \to \infty$
(and dont fit into AdS_5-throat of radius L)
therefore only closed strings can remain
far from the brane.

only closed strings

Maldacena's idea: **subtract the closed strings from both sides** and equate what is left

$$\Rightarrow \quad AdS/CFT \qquad \text{correspondence}$$

For λ large: The very strongly interacting open strings problem on the boundary would thus become replacable by the moderately curved GRT description in the bulk of AdS.

For λ small: The highly curved (large inverse curvature radia) and rather ill-defined (quantum) gravity problem in the bulk becomes replacable by the weakly interacting open strings problem without any gravity on the boundary.

Matching of parameters (following Polchinski):

- gauge theory (parameters g^2_{YM}, N):

$$g^2_{YM} = 4\pi g_S \qquad YM \text{ coupling}$$

$$L^5 \int_{S_5} {}^*F^{(5)} = L^5 \int_{S_5} F_5 = N \qquad \text{integer because of Dirac quantization}$$

- AdS: (parameters L (size of AdS and radius of S_5),
  N number of branes, also integer

$$\underbrace{R_{\mu\nu}}_{\sim L^{-2}} = \underbrace{G_N}_{\sim g_S^2\ \alpha'^4} \underbrace{F^{(5)}{}_{\mu\alpha\beta\gamma\delta}\ F^{(5)}{}_{\nu}{}^{\alpha\beta\gamma\delta}}_{\sim \frac{N^2}{L^{10}}}$$

$$\Rightarrow \frac{L^4}{\alpha'^2} = \underbrace{4\pi g_S}_{g^2_{YM}} N = \lambda \qquad {}'\text{t Hooft coupling}$$

$$L/\alpha'^{1/2} = (4\pi g N)^{1/4} = (g_{YM}^2 N)^{1/4} = (\lambda)^{1/4}$$

! λ very large for classical description !

and gravity parameters:
Define a reduced Planck length $\hat{L}_{P,D}$
such that we have in D dimensions

$$S_{Einstein} = (1/2\hat{L}_{P,D}^{D-2}) \int d^D x R.$$

Then usual Planck length and the reduced one in D=4

$$\hat{L}_{P,4} = (8\pi)^{(1/2)} L_P;$$

In string theory $\hat{L}_{P,10}^8 = (1/2)(2\pi)^7 g^2 \alpha'^4$

and $L/\hat{L}_{P,10} = 2^{-1/4} \pi^{-5/8} N^{1/4}$

! must be large for classical description ! (N very large)

Conjecture: Horowitz, Polchinski [gr-qc/0602037]

Hidden inside 'any' non-abelian gauge theory is a
quantum theory of gravity
i.e. a theory with a massless spin-2 field (graviton)
(in some respects like a composite of the gauge boson)

- A no-go theorem (Weinberg-Witten) seems to forbid this (**QFT forbids massless particles with spin > 1 in non-abelian gauge theories)**.
  But here this is circumvented because
  graviton and QFT live in different spaces

- This meshes well with the **Holographic conjecture**:
  The information of quantum gravity in a given spatial domain can be thought to reside in the boundary of that domain.

  That's because a quantum theory of gravity has
  maximum entropy $\sim$ (D-2)-dimensional area of boundary $\mathcal{A}$

(from Bekenstein/Hawking entropy of black holes $S_{BH} = \frac{\mathcal{A}}{4G_N}$)

Meaning of the extra dimension:

any local QFT has an additional dimension, in which it is local, the energy-scale z.
Coupling constant(s) depend on the energy scale also in a local way via the Callan/Symanzik equation(s)

$$z\frac{\partial g(z)}{\partial z} = \beta(g(z))$$

So r can be interpreted as the inverse (dimensions !) of the energy-scale z

$$r = \frac{1}{z}$$

Since for $r \to 0$ we look at energy-scale $\to \infty$, all finite energy excitations become low energy in comparison (just another way to look at the redshift).

Dictionary of the correspondence:
gauge field versus type II B string theory

- parameters $g_{YM}^2 = 4\pi g_s, \quad g_{YM}^2 N = \frac{L^4}{\alpha'^2}$

- correlation functions of observables (i.e. local gauge-invariant operators $O(\vec{x})$, $\vec{x} =$ spacetime coords of the field-theory)

$$\langle e^{\int d^4\vec{x}\phi_0(\vec{x})O(\vec{x})}\rangle_{CFT} = \mathcal{Z}_{string}\{\phi(\vec{x}, r \to \infty) = \phi_0(\vec{x})\}$$

- Operators $O(\vec{x})$ correspond to bulk-fields $\phi(\vec{x}, r)$ in which the strings propagate, with the same quantum numbers and symmetries, but otherwise remaining rather unspecified.

- The source of $O(\vec{x})$ is the required value of the bulk-field $\phi(\vec{x}, r)$ at the boundary $r \to \infty$

Strongest form of the conjecture:
it holds for all values of g_{YM} and N

A proof is not in sight, and certainly very difficult. But also not disproved yet, even though a single counterexample would suffice!

Two limiting versions are important for applications:

- The 't Hooft limit $N \to \infty$ together with $\lambda = N g_{YM}^2 = 4\pi g_s N$ fixed, i.e. with $g_s \to 0$

- in which case the
planar diagrams of the $SU(N)$ theory become dominant, which resemble the diagrams of a string-theory.

- The Maldacena-limit: $\lambda \to \infty$ after the 't Hooft limit, in which case the curvature radius L of AdS_5 becomes very large and classical super-gravitational theory can effectively replace string theory.

The correspondence under simplifying assumptions

- consider non-Abelian gauge theory $SU(N)$ with many colours, N lar-ge
- To make the new coordinate r or $z = 1/r$ simple, consider scale-invariant case

$$x^\mu \to \lambda\, x^\mu \quad \text{symmetry}$$

on top of Lorentz invariance
$\to$ together they imply conformal invariance.

- work in strong coupling limit $g_s N$ large
- to avoid instability consider susy theory. (Theory with maximal susy in $D = 4 : \mathcal{N} = 4$ SYM. with conformal symmetry $\beta = 0$).

If $\beta = 0$, the coupling is arbitrary and allows to adiabatically change it from weak to strong coupling.

$\mathcal{N}=4$ susy $SU(N)$ CFT in $D=4$
is dual to
type II B string theory on $AdS_5 \times S^5$ with N 3-branes

- effective action, relevant part

$$S \sim \alpha'^{-4} \int d^{10}\, x \, \sqrt{-G}\, \left(e^{-2\phi} R - F_{MNPQR}\, F^{MNPQR}\right)$$

  Take the purely spatial ('magnetic') components of F as independent variables.

- do KK reduction on S_5

$$F_{MNPQR} \sim N\alpha'^2 \qquad \text{from Dirac quantization}$$

$$\int_{S_5} F^{(5)} = N\alpha'^2$$

and integrate over 5 coordinates $\perp \mathrm{AdS}_5$ which are the coordinates on the S_5

$$S_{(5)} \sim \alpha'^{-4} \int d^5x \sqrt{-G_5}\, r^5 \left(e^{-2\phi} R_5 + e^{-2\phi} \frac{1}{r^2} - \frac{\alpha'^4 N^2}{r^{10}} \right).$$

- Perform a conformal rescaling with a suitable scaling factor λ to bring it to Einstein/Hilbert form

$$G^E_{\mu\nu} = \lambda G_{(5)\mu\nu} \quad , \quad \sqrt{|G^E|} = \lambda^{5/2} \sqrt{-G_5} \quad , \quad R^E = \frac{1}{\lambda} R_5 \;;$$

$$\text{want} \quad \sqrt{|G^E|} R^E = \sqrt{-G_5}\, \underbrace{r^5 e^{-2\phi}}_{\stackrel{!}{=} \lambda^{3/2}} R_5 \quad \Rightarrow \quad \lambda = \left(r^{10} e^{-4\phi} \right)^{1/3}$$

Hence, with (later) change of notation $G^E, R^E \to G, R$

$$S_{(5)} \sim \alpha'^{-4} \int d^5x \sqrt{-G^E} \left(R^E \underbrace{+ \frac{e^{-2\phi}}{r^2} \lambda^{-5/2}\, r^5 - \frac{\alpha'^4 N^2}{r^{10}} \lambda^{-5/2}\, r^5}_{-V(r,\phi)} \right)$$

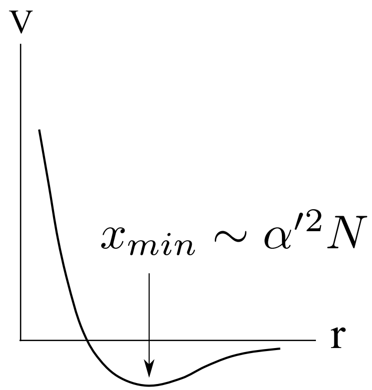
$$\lambda^{-5/2} = \left(r^{10} e^{-4\phi} \right)^{-5/6}$$

$$V(r,\phi) = \alpha'^4 \, N^2 \, e^{\frac{10}{3}\phi} \, r^{-\frac{40}{3}} - e^{+\frac{4}{3}\phi} \, r^{-\frac{16}{3}}$$

$$\sim -x^{-4/3} + \alpha'^4 N^2 x^{-10/3}$$

from flux, dominates at small x

where $x = r^4 \, e^{-\phi}$



minimizing does not fix r and $g_s = e^{\phi}$ separately, but gives negative minimum (AdS !) at

$x_{min} \sim \alpha'^2 N$ Radius of AdS and of S_5 $L^4 = r^4_{min} \sim \alpha'^2 \, N \, e^{\phi}$

Check of holography:

degrees of freedom in $d = 3$ field theory of $N \times N$ matrices
= (# volume-cells δ^3 of lattice used in regularization in box $R^{(3)}$)$\times N^2$
= $\frac{R^3}{\delta^3} N^2$ should be equal to

$$\frac{\text{Area of boundary of } AdS_5}{4G^{(5)}_{Newton}}$$

since

$$(\text{Area of boundary for } \text{z} = \delta \to 0) = \int_{R^{(3)}} d^3x \, \frac{L^3}{z^3} = \frac{R^3 L^3}{\delta^3}$$

and

$$G^{(5)}_{Newton} \sim \frac{L^3}{N^2}$$

both agree,

$$\frac{\text{Area of boundary}}{G^{(5)}_{Newton}} \quad \sim \quad \# \text{ degrees of freedom}$$