AdS/CFT Correspondence
with Applications to Condensed Matter

Robert Graham

UNIVERSITÄT DUISBURG ESSEN

SFB/TR 12: Symmetries and Universality in Mesoscopic Systems

I. Quantum Field Theory and Gauge Theory

II. Conformal Field Theory

III. Short Introduction to Supersymmetry

IV. General Relativity

V. Some String Theory Introduction

VI. A hand-waving derivation of AdS/CFT

VII. Holographic superconductors

Holographic superconductors

G. Horowitz 2010 [hep-th 1002.1722]
with figures taken from G. Horowitz, Holographic superconductors, Milos Lectures 1-2, 2010.

1. Superconductivity

- resistivity drops to 0 below T_c
- magnetic field is expelled below T_c, Meissner effect, perfect diamagnetism

London 1935:

$$J = -\lambda A \quad \Rightarrow \frac{\partial J}{\partial t} \sim -\frac{\partial A}{\partial t} \sim E$$

E-field accelerates charge

$$rot B = 4\pi J = -4\pi\lambda A \quad \Rightarrow \nabla^2 \, B = 4\pi\lambda \, B$$

B-field has finite penetration-depth

Landau, Ginzburg 1950, order parameter complex field ϕ

$$F = \alpha(T - T_c) \, |\phi|^2 + \frac{\beta}{2} \, |\phi|^4 + \xi^2 \, |\nabla\phi|^2 \; ; \;\; \alpha, \beta > 0$$

$$T > T_c \quad \phi = 0 \;\; , \;\; T < T_c \quad |\phi|^2 = \frac{\alpha}{\beta} \, |T - T_c|$$

microscopic theory BCS 1957:

electrons interact via phonons and can form pairs of opposite spin, charged bosons which condense below T_c ;

pairs are loosely bound, and much larger than lattice spacing.

Ground state: energy gap for creation of charged excitation
= dressed electrons and holes

$$\text{energy gap} \quad \Delta \simeq 1.7\, T_c \ .$$

Optical conductivity $\sigma(\omega)$ has a gap

$$\omega_g = 2\Delta \simeq 3.5\, T_c \ .$$

Highest T_c of a BCS-superconductor is about $40°$K $($Mg B_2 $2001)$

High T_c (Bednorz and Müller) 1986
cuprate compounds with CuO_2 planes;
highest T_c for a Hg-Ba-CuO_2 compound

$$T_c = 134^\circ K \ \text{ at normal pressure}$$

$$\sim 160^\circ K \ \text{ at higher pressure}$$

In 2008 Fe instead of Cu,

also in a layered structure

(iron pnictides, with other elements of nitrogen group, like As.)

Again electron pairs, but with unknown pairing mechanism.

The pairing mechanism must come from strong coupling.

AdS/CFT seems to be a good tool to study it.

So far only in models. No connection yet with a microscopic theory.

In his review Horowitz remarks:

2. Gravitational dual model

Want to have an AdS background to describe a strongly coupled field theory by a weakly coupled gravitational field in AdS.

Need a black hole in that background with surface gravity κ, to introduce a finite temperature $T = \kappa/2\pi$.

The black hole should be planar and parallel to the boundary of AdS (translational invariance).

Black holes in AdS: their Hawking temperature increases with their mass $\Rightarrow$ positive specific heat.

Need a “black hole with hair”,
namely with a condensate at small T, which breaks a global $U(1)$.

Idea of S. Gubser, PR D, 78 (2008):

$$S = \int d_x^{D+1} \sqrt{-g} \Big(R + \underbrace{\frac{6}{L_2}}_{\swarrow} \underbrace{- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\searrow} - |\nabla \psi - iqA\psi|^2 \underbrace{- m^2 |\psi|^2}_{\downarrow} \Big)$$

for AdS with black hole, cosm. constant $\wedge = - \frac{3}{L^2}$

U(I)-gauge theory in bulk

scalar 'hair' with mass m, charge q

here

$$m_{eff}^2 = m^2 + q^2 \underbrace{g^{tt}}_{<0} A_t^2$$

may become negative near horizon where $g_{tt} \rightarrow -0$

$\Rightarrow \ \psi = 0$ becomes unstable.

The instability happens as T is lowered, because as $T \rightarrow 0$ the charged Reissner-Nordstrøm b.h. gets closer to extremality $(M \rightarrow Q^+)$, but then $g_{tt} \rightarrow 0$ with double root at horizon, and hence $g^{tt} \rightarrow -\infty$ even faster.

Note: this only works in AdS and not for asymptotically flat black hole.

Then it is not possible to reach extremal black hole because electric field would get too strong before that, and would pull pairs of opposite charges out of vacuum via Schwinger-mechanism; one charge would fall into b.h. and reduce its charge, and the other repelled and escape to infty.

But in AdS, it cannot escape and instead forms 'hair': a gas above the black hole outside its horizon = the charged condensate.

For $D+1=4$ the bulk-theory is dual to a 2+1-dimensional boundary theory (e.g. for describing the superconducting layers in the cuprates or the pnictides).

For $D+1=5$ the boundary theory describes 3d superconductors.

Limitation: can so far only describe global U(I) symmetry breaking in boundary theory;
electromagnetic field on boundary is external, like in BCS, and not dynamical. **Horowitz:**

3. The probe limit (i.e. neglecting backreaction on geometry)

rescale

$$A_\mu = \tilde{A}_\mu / q \;\; , \;\; \psi = \tilde{\psi}/q \; ; \quad \text{then action} \quad S \sim \frac{1}{q^2}$$

$$q \to \infty \;\; , \;\; \tilde{A}, \tilde{\psi} \;\; \text{fixed,} \qquad \text{defines the probe limit.}$$

Drop the $\sim$ hence forth

Consider case D+1=4. Planar Schwarzschild-AdS b.h.

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(dx^2 + dy^2\right)$$

$$AdS_4 : \quad f(r) = \frac{r^2}{L^2}$$

$$AdS_4 - \text{Schwarzschild} \quad f = \frac{r^2}{L^2}\left(1 - \frac{r_0^3}{r^3}\right)$$

$$\text{Hawking temperature} \quad T = \frac{3r_0}{4\pi L^2}$$

In probe limit this is a <u>fixed background in which to find</u> $\underline{A, \psi}$.

The condensate:

take $A_r = A_x = A_y = 0$, only $A_t = \phi \neq 0$

ψ then has const. phase, take ψ real

$$\psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi' + \frac{\phi^2}{f^2}\psi - \frac{m^2}{f}\psi = 0 \qquad (1)$$

$$\phi'' + \frac{2}{r}\phi' - \frac{2\psi^2}{f}\phi = 0 \qquad (2)$$

- Take m^2 as the $mass^2$ of conformally coupled scalar in AdS_4

$$m^2 = -\frac{2}{L^2} > m^2_{BF} = -\frac{D^2}{4L^2}$$

This is negative but allowed in AdS_4 because it is above the BF bound in AdS_4 ,

$$m^2_{BF}(AdS_4) = -\frac{9}{4L^2} \ .$$

- Boundary condition at $\underline{r = r_0}$:

$A_0 = 0$ for $g^{\mu\nu} A_\mu A_\nu$ to remain finite;

(or $g^{\mu\nu} A_\nu \sim j^\mu$ to stay finite,

or from eq. (2) at $r = r_0$, where $f(r_0) = 0$)

$f'\psi' = m^2\psi$ (from eq. (1) at $r = r_0$)

Boundary condition at $r \to \infty$ (boundary of AdS_4)

asymptotics of solution

$$\psi = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + \dots$$

$$\phi = \mu - \frac{\rho}{r^2}$$

'standard' b.c.: $\psi^{(1)} = 0$, $\psi^{(2)} \neq 0$, $\psi \sim \frac{1}{r^2}$ near boundary

'nonstandard' b.c.: $\psi^{(2)} = 0$, $\psi^{(1)} \neq 0$, $\psi \sim \frac{1}{r}$ is also possible.

Numerical solution in Hartnoll, Herzog, Horowitz, PRL 101 (2008).

What is the dual CFT on the boundary?

- $D = 2+1, \quad T = \frac{3r_0}{4\pi L^2}$;
- chemical potential μ ;
- charge density ρ ($\neq 0$!! , does not include a compensating background charge)
- contains a scalar operator $\mathcal{O}$ with global U(1)-symmetry, dual to ψ:

  standard b.c. $\psi^{(1)} = 0\,,\ \psi^{(2)} \neq 0$ on boundary
  corresponds to operator $\mathcal{O}_2 = \psi^{(2)}$ with scaling dimension 2

  nonstandard b.c. $\psi^{(2)} = 0\,,\ \psi^{(1)} \neq 0$
  corresponds to $\mathcal{O}_1 = \psi^{(1)}$ with scaling dimension 1.

4. Scaling symmetry of CFT at finite T:

if $T \to aT$ — a is scaling parameter

then $r \to ar, \quad r_0 \to ar_0$

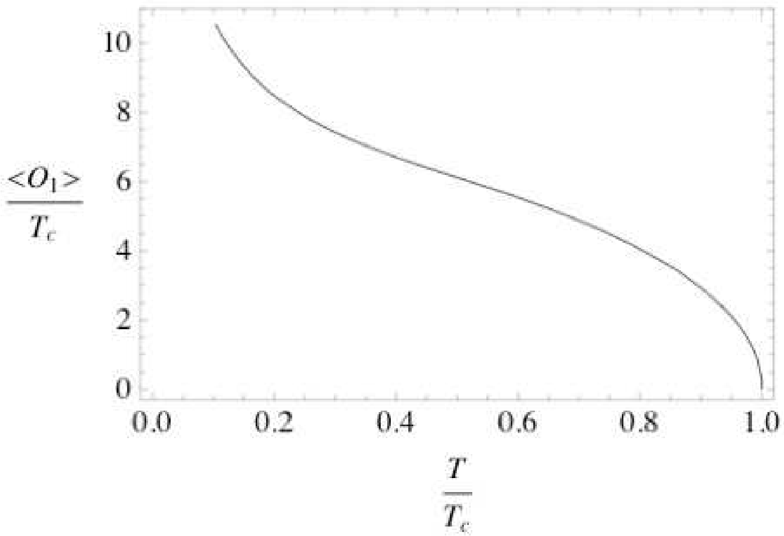
and

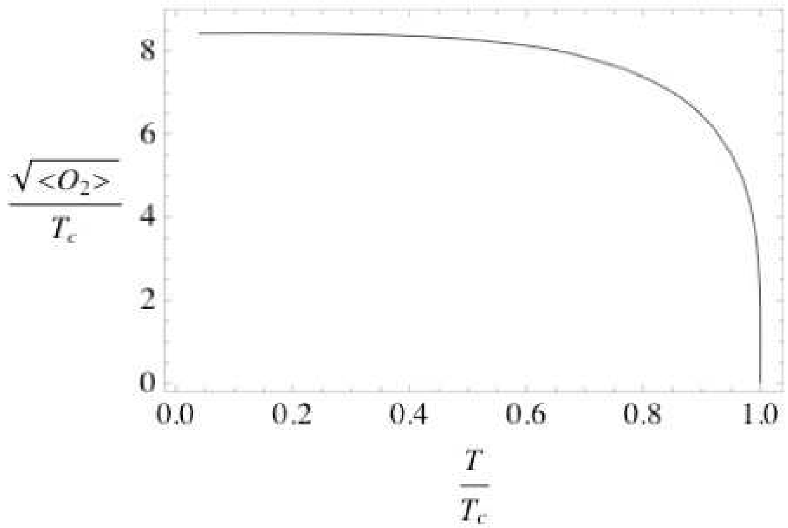
$$
\begin{aligned}
(t,x,y) &\to (t,x,y)/a \\
f &\to a^2 f \\
\phi &\to a\phi \ , \ \mu \to a\mu \ ; \ \rho \to a^2\rho. \\
\mathcal{O}_2 &\to a^2\mathcal{O}_2 \ , \\
\mathcal{O}_1 &\to a\mathcal{O}_1
\end{aligned}
$$

μ or T_c can be used to fix a scale.

$\frac{\sqrt{\mathcal{O}_2}}{\mu}$ plotted as function of $\left(\frac{T}{\mu}\right)$
it turns out $\neq 0$ only below a T_c.

O_i has dimension i, and μ has dimension one,
so O_i / T^i and T/μ are dimensionless.





$$T_c \propto \mu$$

Then it is convenient to use T_c (which turns out $\sim \mu$) to fix a scale

Behavior near T_c from numerical result

$$< \mathcal{O}_1 > \simeq 9.3\, T_c\, (1 - T/T_c)^{1/2}$$

$$< \mathcal{O}_2 > \simeq (12\, T_c)^2\, (1 - T/T_c)^{1/2}$$

Free energy (= Euclidean action). Is smaller than Euclidean action

at $\psi = 0\,,\ \phi = \mu - \frac{\rho}{r}$

and becomes equal for $T \to T_c$

$\Longrightarrow$ second order phase transistion.

5. Arbitrary mass for field ψ

In AdS_{D+1} with $z = -\frac{L^2}{r}$

$$ds^2 = L^2 \, \frac{dz^2 + dx^\mu dx_\mu}{z^2} = g_{AB} \, dx^A \, dx^B$$

$$A = 0 \dots D \quad , \quad x^A = (z, x^\mu) \quad , \quad \sqrt{|g|} = \frac{L^2}{z^2}$$

$$S = -\frac{\kappa}{2} \int d^{D+1}x \sqrt{|g|} \left[g^{AB} \partial_A \, \psi \, \partial_B \, \psi + m^2 \psi^2 + \mathcal{O}(\psi^3) \right]$$

by partial integration with respect to z from horizon $z = -\infty$ to boundary $z = -\epsilon$ and using Stokes' theorem

$$= -\frac{\kappa}{2} \int\limits_{\partial(AdS)} d^D x \sqrt{|g|} \; g^{zB} \; \psi \; \partial_B \psi - \underbrace{\frac{\kappa}{2} \int d^{D+1}x \sqrt{|g|} \; \psi \Big(-\Box + m^2 + \mathcal{O}(\psi^2) \Big) \psi}_{=0 \text{ on shell}}$$

only the boundary term remains.

Translational invariance along boundary

$$\psi\,(z,x^\mu) = e^{ik_\mu x^\mu} f_k(z)$$

$$\Rightarrow \quad \left(g^{\mu\nu} k_\mu k_\nu - \frac{1}{\sqrt{|g|}}\,\partial_z\,\sqrt{|g|}\,g^{zz}\,\partial_z + m^2\right) f_k(z) = 0$$

solved by Bessel functions.

Near $\quad z \to 0 \quad : \quad f_k = z^\Delta$

$$0 = \underbrace{k^2\;z^{\Delta+2}}_{\to 0} \;-\; z^{D+1}\,\Delta\,(\Delta - D)\,z^{\Delta-D-1} + m^2\,L^2\,z^\Delta$$

$$m^2 L^2 = \Delta\,(\Delta - D)$$

$$\Delta_\pm = \frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + m^2\,L^2}$$

In AdS $\;m^2 < 0\;$ need <u>not</u> be a tachyon.

But if $\Delta_\pm$ must be real, then

$$m^2 > m_{BF}^2 = -\frac{D^2}{4L^2}$$ Breitenlohner Freedman bound

- $\Delta_+ > 0 \quad \Rightarrow \quad z^{\Delta_+} \to 0$ at boundary
- $\Delta_- < \Delta_+ \quad \Rightarrow \quad z^{\Delta_-}$ dominates over z^{Δ_+} for $z \to 0$
- $\Delta_+ + \Delta_- \quad = \quad D$
- standard boundary condition (cutoff ε)

at $z = \varepsilon :$ $$\psi(x,z) = \underbrace{\varepsilon^{\Delta_-}}_{\text{singular}} \; \underbrace{\psi^{ren}(x)}_{\text{finite for } \varepsilon \to 0}$$

6. Renormalization of dual boundary operator $\mathcal{O}$ to which $\psi(x,\varepsilon)$ couples

$$S_{\text{bound}} = \cdots + \int d^D x \sqrt{|\gamma_\varepsilon|} \; \underbrace{\psi(x,\varepsilon)\, \mathcal{O}(x,\varepsilon)}_{\text{joint scaling dimension D}}$$

metric on boundary

$$ds^2|_{z=\varepsilon} = \gamma_{\mu\nu}(\varepsilon)\, dx^\mu\, dx^\nu = \frac{L^2}{\varepsilon^2}\, \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$\Rightarrow \sqrt{|\gamma|} = \left(\frac{L}{\varepsilon}\right)^D$$

For standard b.c. near $z = 0$:

$$S_{\text{bound}} = \cdots + \int d^d\, x \underbrace{\left(\frac{L^D}{\varepsilon}\right) \varepsilon^{\Delta_-}\, \mathcal{O}(x,\varepsilon)}_{\text{must be finite for } \varepsilon \to 0}\; \psi^{ren}(x)$$

$$\Rightarrow \quad \mathcal{O}(x,\varepsilon) \sim \varepsilon^{-\Delta_- + D} = \varepsilon^{\Delta_+}\, \mathcal{O}^{ren}$$

i.e. the scaling dimension of $\mathcal{O}^{ren}$ is Δ_+, of ψ it is Δ_-.

choosing nonstandard b.c. near $z = 0$:

$$\psi(x,z) = \varepsilon^{\Delta_+}\, \psi^{ren}(x)$$ has now dimension Δ_+

then $$S_{\text{bound}} = \cdots + \int d^D x \underbrace{\left(\frac{L^D}{\varepsilon}\right) \varepsilon^{\Delta_+}\, \mathcal{O}(x,\varepsilon)}_{\text{finite! for } \varepsilon \to 0}\ \psi^{ren}(x)$$

$$\mathcal{O}(x,\varepsilon) \sim \varepsilon^{-\Delta_+ + D}\, \mathcal{O}^{ren} = \varepsilon^{\Delta_-}\, \mathcal{O}^{ren}$$

i.e. the scaling dim. of $\mathcal{O}^{ren}$ then is Δ_-.

A second boundary condition for ψ is required:

ψ regular for $z \to -\infty$

or for $z \to$ horizon (of the b.h. setting the temperature, e.g. 'ingoing' b.c.)

Note:

- $\Delta_\pm$ independent of k and x (typical for a local QFT)

- if $m^2 > 0$ $\Delta_+ > D$, $\mathcal{O}$ then "irrelevant"
  irrelevant means getting weak towards infrared, growing strong towards ultraviolet

- if $m^2 < 0$, then $\Delta_+ < D$, $\mathcal{O}$ then "relevant"
  i.e. growing strong in infrared, growing weak in ultraviolet;
  $\psi \sim z^{\Delta_\pm}$ then both decay towards boundary

- if $m^2 = 0 \longleftrightarrow \Delta_+ = D$, $\Delta_- = 0$, $\mathcal{O}$ then "marginal"

Applied to AdS_4, with mass m for scalar ψ:

$$\text{for} \quad r \to \infty : \quad \psi = \frac{\psi_-}{r^{\Delta_-}} + \frac{\psi_+}{r^{\Delta_+}}$$

$$\Delta_\pm = \frac{1}{2}\left(3 \pm \sqrt{9 + 4(mL)^2}\right)$$

- if $m^2 \geq m_{BF}^2 + \frac{1}{L^2}$ only solution with $\psi_- = 0$ normalizable

$$\psi_+ \longleftrightarrow \mathcal{O} \quad \text{with dimension} \quad \Delta_+$$

$$\psi_- \longleftrightarrow \quad \text{source for} \quad \mathcal{O}$$

want spontaneous condensation of $\mathcal{O}$, hence set its source to 0
$\Rightarrow$ must choose standard b.c.$\psi_- = 0$

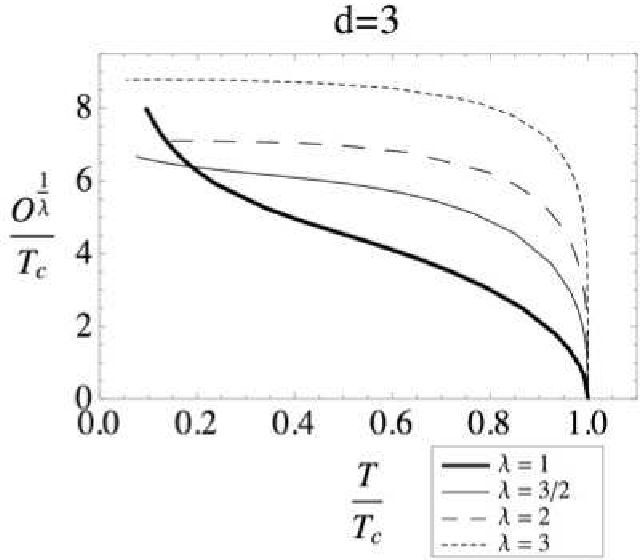
- if $m_{BF}^2 \leq m^2 < m_{BF}^2 + \frac{1}{L^2}$ also the solution with ψ_- is normalizable; alternate (nonstandard) b.c. $\psi_+ = 0$ becomes possible. Then

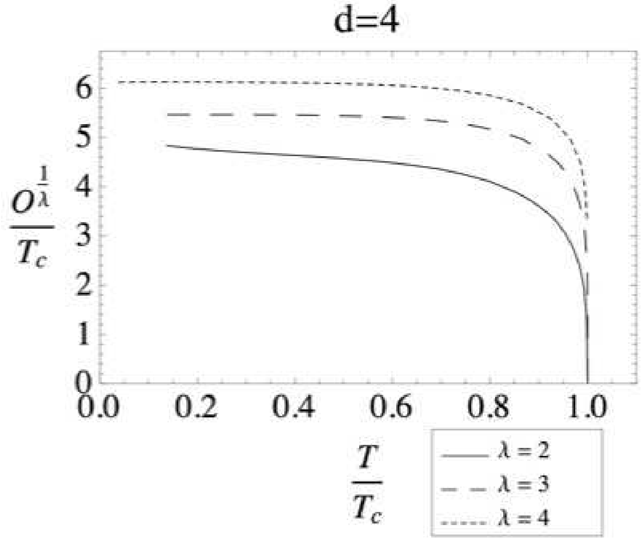
$$\psi_- \longleftrightarrow \mathcal{O} \quad \text{with dimension} \quad \Delta_-$$

$$\psi_+ \longleftrightarrow \quad \text{source for} \quad \mathcal{O}$$

so again condensation is spontaneous.

Condensate (hair) as a function of T





$$T_c \propto \mu$$

here λ is the same as Δ in our notation

note:

the curves $\lambda = 1$, $\lambda = 2$ both correspond to $(mL)^2 = -2$, but different b.c.:

$\lambda = 1$: alternate

$\lambda = 2$: standard

all other curves are for standard b.c..

The curve

- $\lambda = 3$ corresponds to $m^2 = 0$
- $\lambda = \frac{3}{2}$ to $m^2 = m^2_{BF} = -\frac{9}{4L^2}$

7. Calculation of optical conductivity:

Take conductivity in arbitrary x-direction

probe field $A_x(r)e^{-i\omega t}$ travelling in y-direction

$$A_x'' + \frac{f'}{f} A_x' + \left(\frac{\omega^2}{f^2} - \frac{2\psi^2}{f}\right) A_x = 0$$

ingoing b.c. at horizon of AdS influences bulk causally and gives retarded Green's function.

Near boundary $A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r} + \ldots$

from the gauge/gravity dictionary:

Electric field on boundary = boundary limit of electric field in bulk

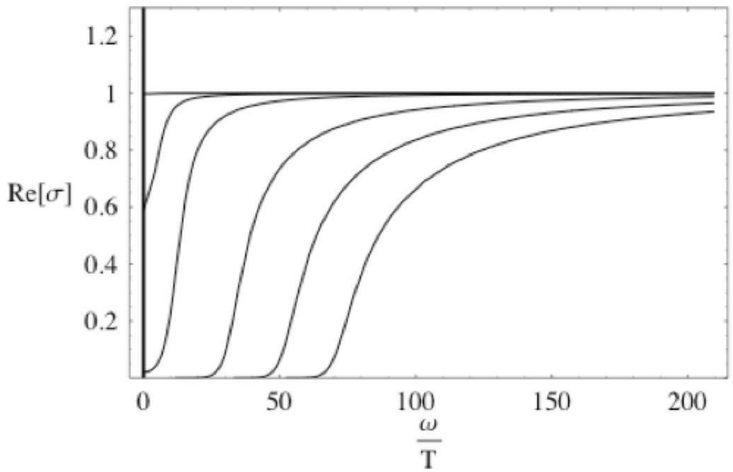
$$E_x = -\left.\frac{\partial A_x}{\partial t}\right|_{r\to\infty} = -\dot{A}_x^{(0)}$$

Induced current $J_x = A_x^{(1)} \quad \Rightarrow \quad \sigma(\omega) = \frac{J_x}{E_x} = \frac{J_x}{+i\omega A_x^{(0)}} = \frac{-iA_x^{(1)}}{\omega A_x^{(0)}}$

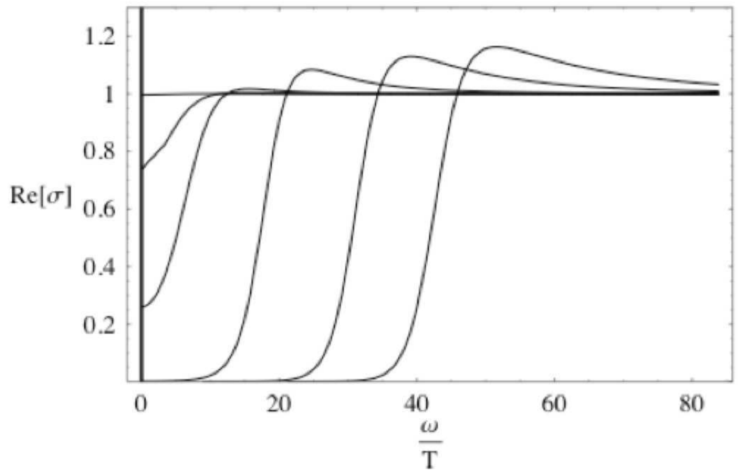
Re $\sigma(\frac{\omega}{T_c})$ for successively lower temperatures

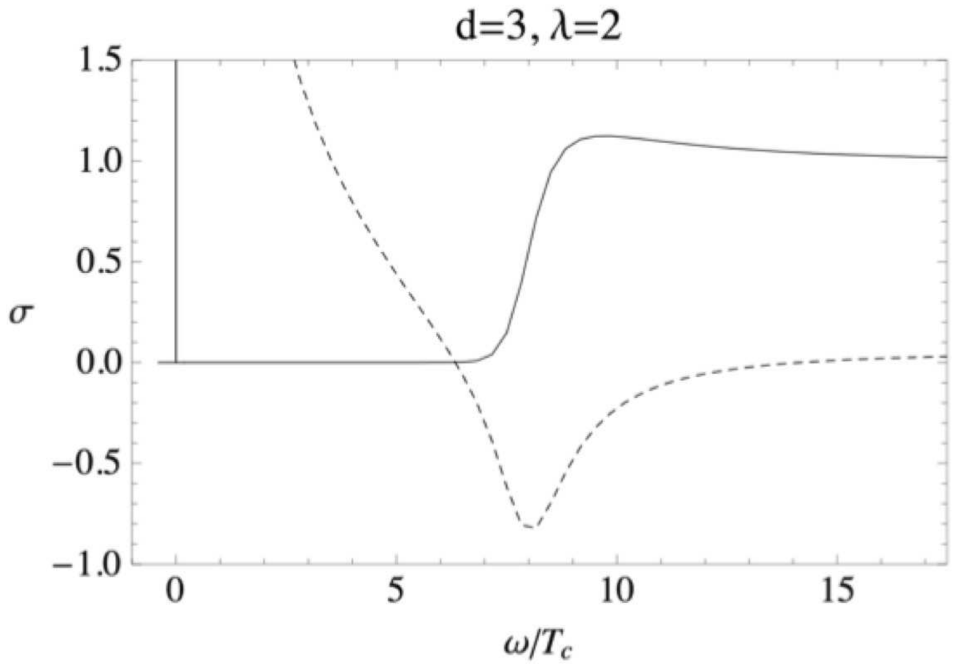
Consider first λ = 1, 2 (d = 3)

$\mathcal{O}_1$



$\mathcal{O}_2$





Low temperature limit of conductivity

Horowitz and Roberts, Phys.Rev. D78 (2008)

Note: nonvanishing dissipation appears.

- The b.h. acts as a heat-bath, since at its horizon it swallows incoming e.m. waves from boundary
- $\frac{\omega_g}{T_c} \simeq 8 > 2\frac{\omega_g}{T_c}\Big|_{BCS}$, measured roughly in high T_c supercond.

In his review Horowitz remarks on this:

For $T < T_c$:

Im $\sigma(\omega)$ has a pole at $\omega = 0$,

(follows already from $\vec{j} = ne\,\vec{v} = \sigma\,\vec{E}$

$$\text{Drude model} \quad m\frac{dv}{dt} = eE - m\frac{v}{\tau} \;,$$

$$\text{i.e.} \quad \sigma(\omega) = \frac{ne^2}{m}\,\frac{\tau}{1 - i\omega\tau} \;;$$

$$\tau \to \infty \;:\; \text{Im}\,\sigma = \frac{ne^2}{m\omega} \;,\quad \text{Re}\,\sigma = \frac{\pi n e^2}{m}\,\delta(\omega)$$

is implied by Kramers-Kronig relation

$$\text{Im}\,\sigma(\omega) = \mathcal{P}\int_{-\infty}^{+\infty}\frac{d\omega'}{\pi}\,\frac{\text{Re}\,\sigma(\omega')}{\omega - \omega'} \quad)$$

Re $\sigma(\omega')$ must therefore contain $\delta(\omega')$

8. London equation:

choose Coulomb gauge in boundary, $\vec{\nabla} \cdot A = 0$

Solve equations for $A_x(r)$ and ψ for a wave travelling in y-direction with r-dependent amplitude

$$A_x = A_x(r) e^{-i\omega t + iky}$$

$$(f A_x')' + \left(\frac{\omega^2}{f^2} - \underbrace{\frac{k^2}{r^2}}_{\text{only change}} \right) A_x = 2\psi^2 A_x$$

asymptotic for $r \to \infty$

$$A_x(\omega, k) = A_x^{(0)}(\omega, k) + \frac{A_x^{(1)}(\omega, k)}{r} + \ldots$$

$$\vec{k} \cdot A^{(0)} = 0 = \vec{k} \cdot A^{(1)} \qquad \text{(Coulomb-gauge)}$$

retarded Green's function for $J_x = A_x^{(1)}$ in dual field theory

$$G^R(\omega, k) = \frac{A_x^{(1)}(\omega, k)}{A_x^{(0)}(\omega, k)} = i\omega\sigma(\omega, k)$$

In boundary

$$J_x(\omega, k) = A_x^{(1)}(\omega, k) \quad , \quad E_x = i\omega A_x^{(0)}(\omega, k)$$

our earlier calculation showed that Im $\sigma(\omega)$ has a pole at $\omega = 0$:

$$\text{Im}\ \sigma(\omega) = \frac{n_s}{\omega} \quad + \text{regular}\ ;$$

Kramers-Kronig relation then implies:

$$\text{Re}\ \sigma(\omega) = +\pi\ n_s \delta(\omega) \quad + \text{regular}$$

$\Rightarrow$ for $\omega, k \to 0$ $\quad \underbrace{A_x^{(1)}}_{\updownarrow}(\omega, k) = -n_s \underbrace{A_x^{(0)}}_{\updownarrow}(\omega, k)$

i.e. in boundary $\quad J_x\ (\omega, k) = -n_s\ A_x(\omega, k)$ London eqn.

Further results for holographic superconductor
(Herzog at al., Horowitz et al.)

From retarded Green's function

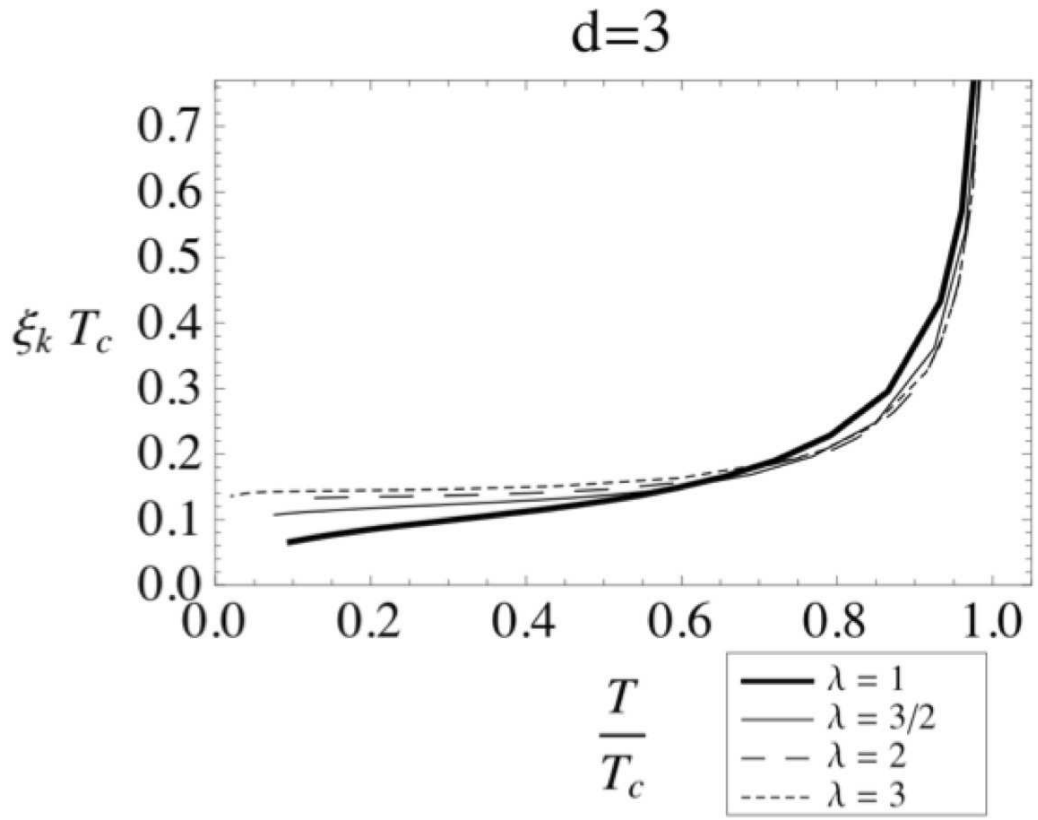
$$G_R(\omega, k) = \frac{A^{(1)}(\omega, k)}{A^{(0)}(\omega, k)}$$

by expansion in k^2 :

$$\text{Im}\, G_R(\omega, k) = -n_s\,(1 + \xi_k^2\, k^2 + \dots)$$

Near T_c

$$\xi\, T_c \simeq \frac{0.1}{(1 - T/T_c)^{1/2}}$$

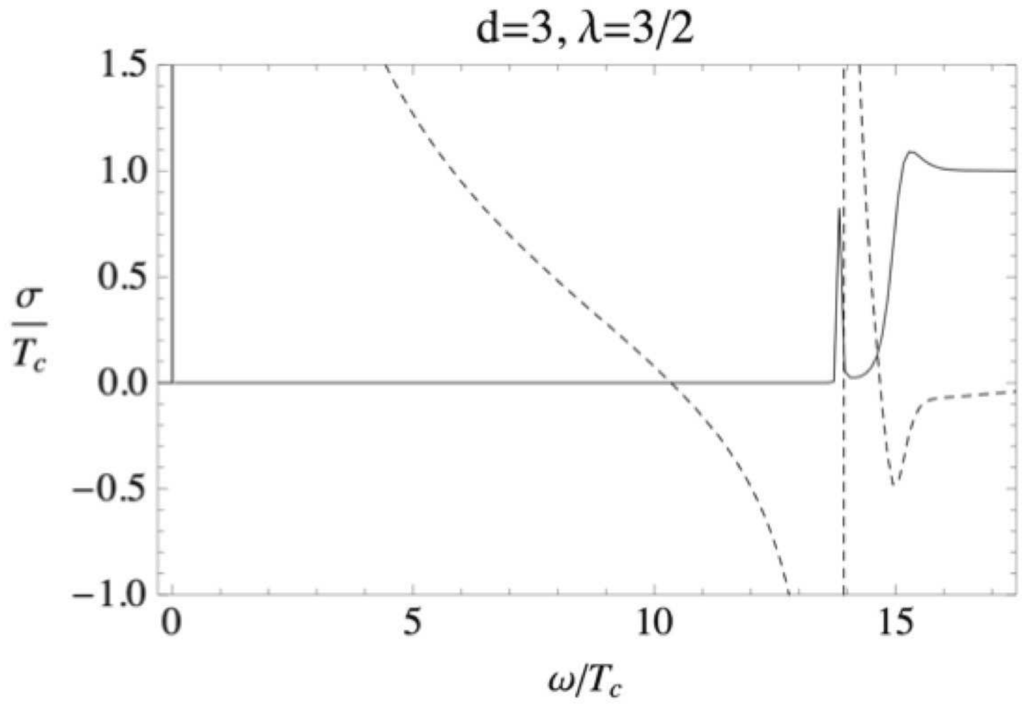


The d=3 correlation length. Near T = T_c:

$$\xi_k T_c = \frac{0.1}{(1 - T/T_c)^{1/2}}$$

If the BF bound is saturated new spikes of $\mathrm{Re}\sigma(\omega)$ appear inside the energy gap;

maybe a new bound state of quasiparticles?



As you lower T, a new spike appears

Beyond the probe limit

The Lagrangian again

$$\mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |\psi - iqA\psi|^2 - m^2|\psi|^2$$

Ansatz with back-reaction

$$ds^2 = -g(r)\, e^{-\chi(r)}\, dt^2 + \frac{dr^2}{g(r)} + r^2(dx^2 + dy^2)$$

$$A = \phi(r)\, dt \quad , \quad \psi = \psi(r)$$

4 coupled ODE's for g, χ, ϕ, ψ

Two scaling symmetries:

1) $t \to at \;\; , \;\; e^{\chi} \to a^2 e^{\chi} \;\; , \;\; \phi \to \phi/a$

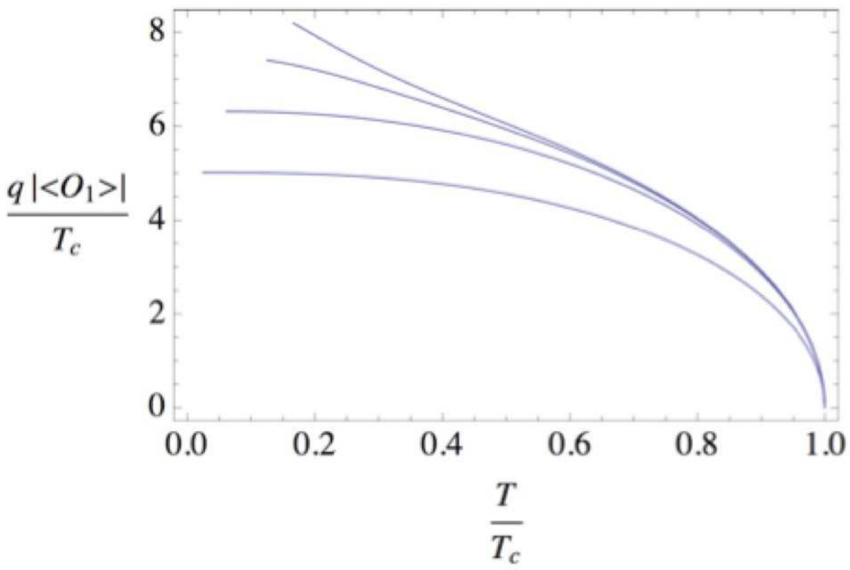
2) $r \to ar \;\; , \;\; (t,x,y) \to \frac{1}{a}(t,x,y) \;\; , \;\; g \to a^2 g \;\; , \;\; \phi \to a\phi$

from 1) : set $\chi = 0$ for $r \to \infty$

from 2) : set $r_0 = 1$ $($provided $T \neq 0)$

Two main differences in numerical solution

- A divergence in the probe limit of $\frac{<\mathcal{O}_1>}{T_c}$ for $T \to 0$ disappears now



From bottom to top, the curves correspond to q = 1, 3, 6, 12

- For m^2 close to BF bound, T_c remains nonzero even when charge of condensate $q = 0$.

  New instability of near extremal charged AdS b.h. to forming neutral scalar hair.

The reason of new instability:

Extremal AdS b.h. has near horizon geometry $AdS_2 \times R^2$

So there are two AdS spaces now,

the original AdS_{D+1} with $m_{BF}^2 = -\frac{D^2}{4}$

and the new AdS_2 with $\left(m_{BF}^2\right)_{\text{new}} = -\frac{1}{4}$.

The new instability (with unclear interpretation in the CFT) appears if

$$-\frac{D^2}{4} = m_{BF}^2 < m^2 < \left(m_{BF}^2\right)_{\text{new}} = -\frac{1}{4}$$

Conductivity beyond probe limit Hartnoll et al. JHEP 0812, 015 (2008)

New equation for A_x (including back-reaction χ)

$$A_x'' + \left(\frac{g'}{g} - \frac{\chi'}{2}\right) A_x' + \left[\left(\frac{\omega^2}{g^2} - \frac{\phi'^2}{g}\right) e^{\chi} - \frac{2q^2\,\psi^2}{g}\right] A_x = 0$$

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r}$$

$$\sigma(\omega) = -\frac{i}{\omega}\,\frac{A_x^{(1)}}{A_x^{(0)}}$$

as before.

A more transparent way to get the conductivity

New radial variable

$$dz = \frac{e^{\chi/2}}{g}\, dr$$

Near horizon (where $g \sim r - r_0 \to 0$) $\quad z \sim \log\,(r - r_0)$

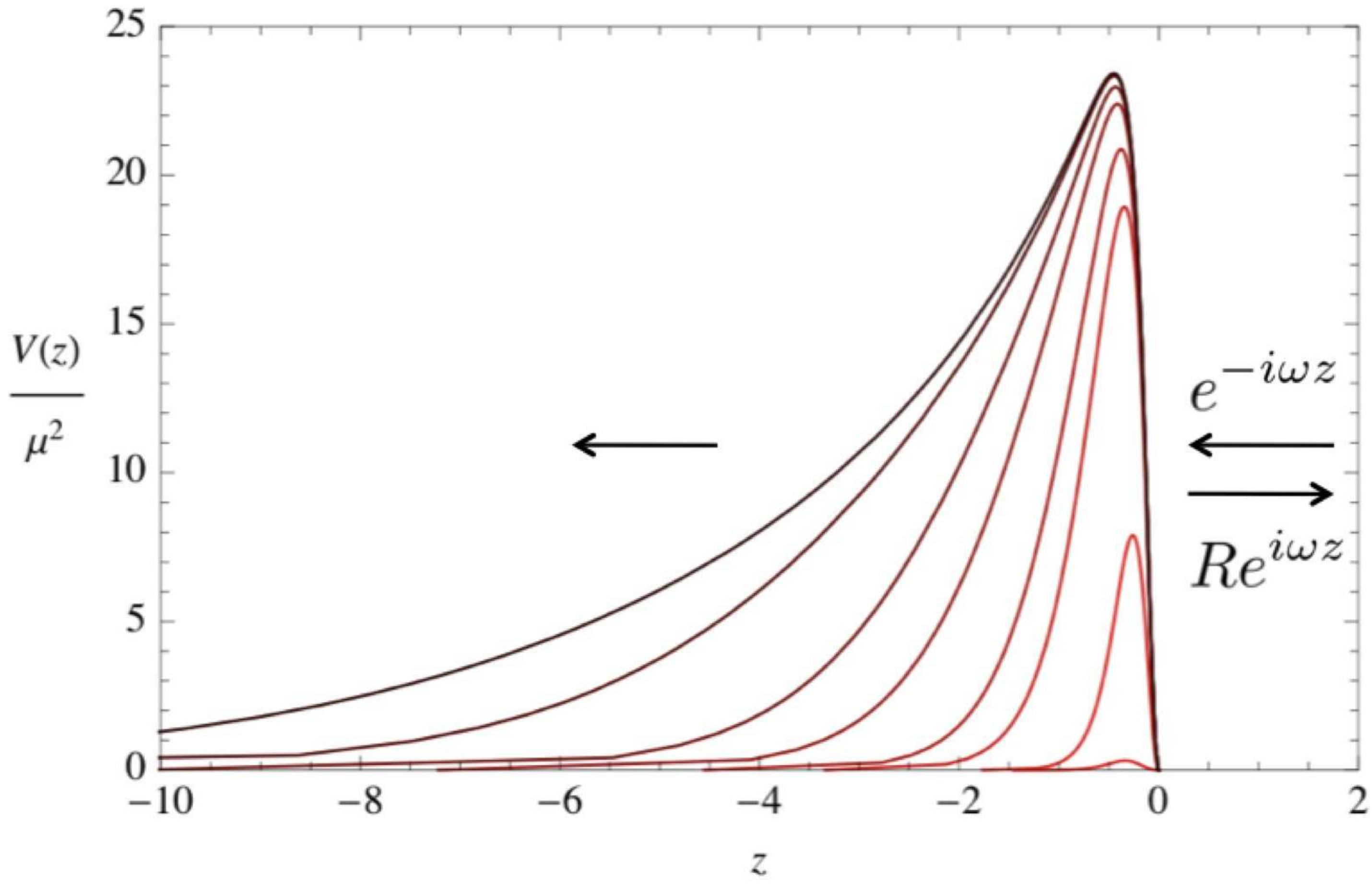
Asymptotically $r \to \infty \quad z = -\frac{1}{r}$

Equation for A_x now looks like 1-d Schrödinger equation

$$-\partial_z^2\, A_x + V(z) A_x = \omega^2 A_x$$

potential $\quad V(z) = g\left[(\partial_r\, \phi)^2 + 2q^2\, \psi^2\, e^{-\chi}\right]$

- at horizon $\quad r = r_0 \quad z \to -\infty \quad : \quad V \to 0$ exponentially
- asymptotically $\quad z \to 0 \quad : \quad V \sim z^{2(\Delta-1)}$



The potential grows as T/T_c gets smaller
(for q= 10, λ = 2)

ω^2 is the incident energy in the Schrödinger-like description. Gap-size $\omega_g^2 \sim V_{max}$

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r}$$

$$\sigma(\omega) = -\frac{i}{\omega}\frac{A_x^{(1)}}{A_x^{(0)}}$$

now $$A_x^{(0)} = A_x|_{z=0} = 1 + R \ ,$$

$$A_x^{(1)} = -\partial_z A_x|_{z=0} = i\omega(1-R)$$

hence $$\sigma(\omega) = \frac{1-R}{1+R}$$

If $A_x^{(0)}$, $A_x^{(1)}$ both real for $\omega \to 0$

$\Rightarrow$ Im $\sigma(\omega) \sim 1/\omega$ has pole, Re $\sigma(\omega) \sim \delta(\omega)$

Reality is easily shown for $\omega = 0$.

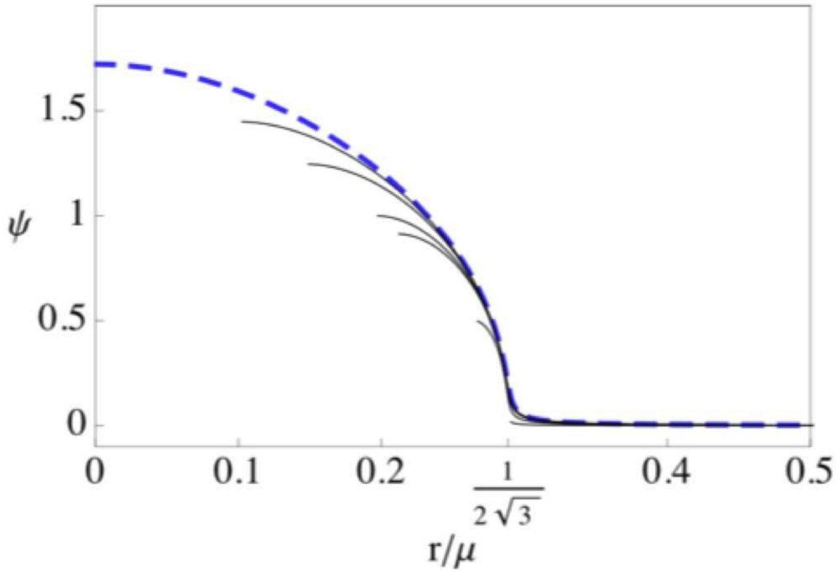
Explanation of spikes in Re σ at $\omega \neq 0$:
Destructive interference between incoming and scattered wave can make $A_x^{(0)}$ very small.
This condition can be evaluated e.g. in WKB as

$$\int_{-z_0}^{0} \sqrt{\omega^2 - V(z)} dz + \pi/4 = n\pi \quad \text{with} \quad \omega^2 = V(-z_0)$$

The $T \to 0$ limit in the holographic superconductor. (Horowitz and Roberts JHEP 0911, 015 (2009))

It is not similar to extremal Reissner-Nordstrøm b.h. (which has large entropy for $T = 0$).
Instead it has vanishing horizon area (entropy) $\frac{r_0}{\mu} \to 0 \quad as \quad T \to 0$ and it has vanishing charge (as long as $q \neq 0$).

Solution not smooth at $r = 0$, near horizon behavior depends on m, q.



Low temperature solutions approach T=0 solution.

$V(z) = 0$ at horizon $z \to -\infty$ still holds

hence there is always a transmitted wave in the scattering description,

and Re $\sigma(\omega) \neq 0$ for small ω also at $T = 0$ and

there is never a sharp gap, even at $T = 0$.

$$V(z) \sim \frac{1}{z^2} \quad \text{near horizon}$$

$$\int_{-\infty}^{0} \sqrt{V(z)}\, dz = \infty$$

$$\lim_{\omega \to 0} \text{Re}\, \sigma(\omega)\Big|_{T=0} = 0$$

Adding magnetic fields

If $($Work to expell applied magnetic field B$) > F_{\text{normal}} - F_{\text{supercond.}}$
then superconductivity is destroyed:

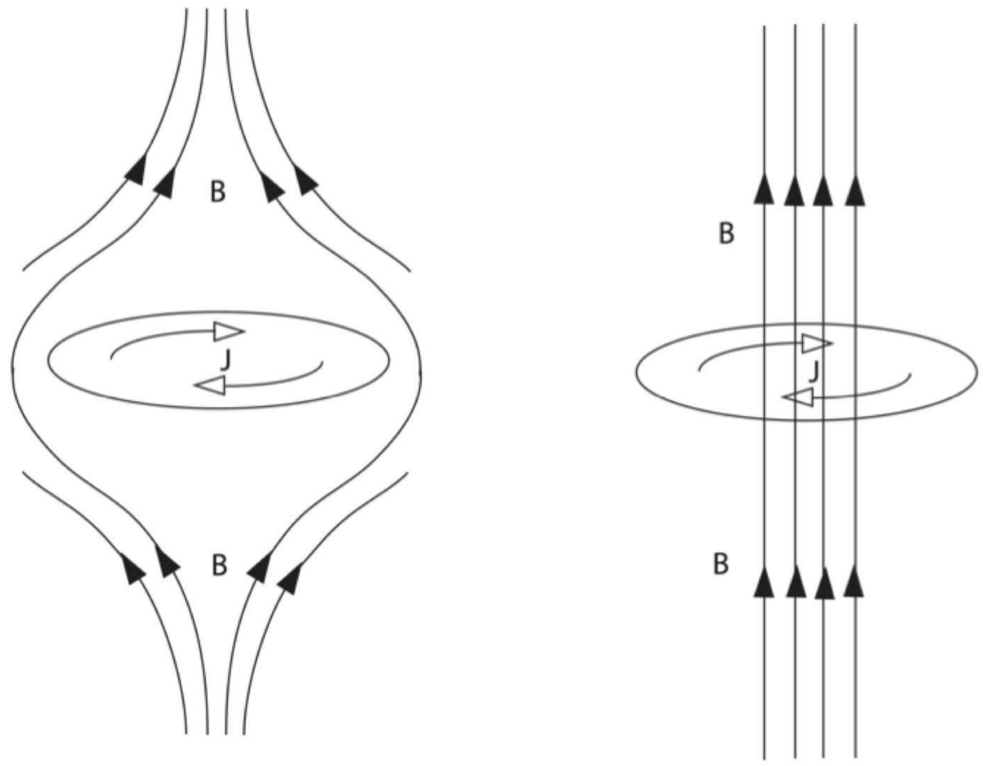
$$\frac{B_c^2(T)V}{8\pi} = F_n(T) - F_{sc}(T)$$

First order transition at $B = B_c$ type I superconductor

second order transition at $B = B_c$: type II

The holographic superconductor turns out as type II.

In D= 3 = 2+1 dimensions $B_c(T) = 0$.



To expel B from disk of radius R, the superconductor must do work ~ R^3. The difference in free energy is only ~ R^2.