RANDOM MATRIX THEORY IN PHYSICS

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Introduction

We wish to study energy correlations of quantum spectra. Suppose the spectrum of a quantum system has been measured or calculated. All levels in the total spectrum having the same quantum numbers form one particular *subspectrum*. Its energy levels are at positions $x_n,\ n = 1, 2, \ldots, N$, say. We assume that N, the number of levels in this subspectrum, is large. With a proper *smoothing* procedure, we obtain the level density $R_1(x)$, i.e. the probability density of finding a level at the energy x. As indicated in the top part of **Figure 1**, the level density $R_1(x)$ increases with x for most physics systems. In the present context, however, we are not so interested in the level density. We want to measure the spectral correlations independently of it. Hence, we have to remove the level density from the subspectrum. This is referred to as *unfolding*. We introduce a new dimensionless energy scale ξ such that $d\xi = R_1(x)dx$. By construction, the resulting subspectrum in ξ has level density unity as shown schematically in the bottom part of **Figure 1**. It is always understood that the energy correlations are analyzed in the *unfolded subspectra*.

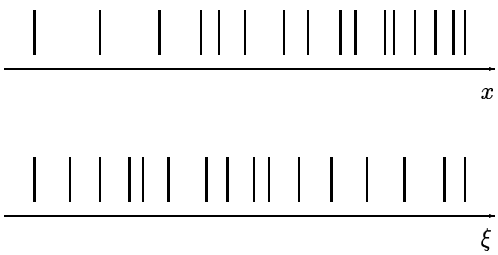

FIG. 1. Original (top) and unfolded (bottom) spectrum.

Surprisingly, a remarkable *universality* is found in the spectral correlations of a huge class of systems, including nuclei, atoms, molecules, quantum chaotic and disordered systems and even quantum chromodynamics on the lattice. Consider the *nearest neighbor spacing distribution* $p(s)$. It is the probability density of finding two adjacent levels in the distance s. If the positions of the levels are uncorrelated, the nearest neighbor spacing distribution can be shown to follow the *Poisson law*

$$p^{(\mathrm{P})}(s) = \exp(-s) \ . \tag{1}$$

While this is occasionally found, many more systems show a rather different nearest neighbor spacing distribution, the *Wigner surmise*

$$p^{(\mathrm{W})}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right) \ . \tag{2}$$

As shown in **Figure 2**, the Wigner surmise excludes degeneracies, $p^{(\mathrm{W})}(0) = 0$, the levels repel each other. This is only possible if they are correlated. Thus, the Poisson law and the Wigner surmise reflect the absence or the presence of energy correlations, respectively.

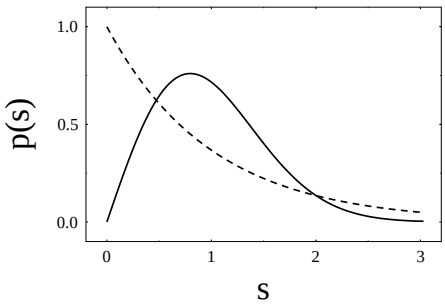

FIG. 2. Wigner surmise (solid) and Poisson law (dashed).

Now, the question arises: If these correlation patterns are so frequently found in physics, is there some simple, phenomenological model? — Yes, Random Matrix Theory (RMT) is precisely this. To describe the absence of correlations, we choose, in view of what has been said above, a diagonal Hamiltonian

$$H = \mathrm{diag}\,(x_1, \ldots, x_N) \ , \tag{3}$$

whose elements, the eigenvalues x_n, are uncorrelated random numbers. To model the presence of correlations, we insert off–diagonal matrix elements,

$$H \ = \begin{bmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & & \vdots \\ H_{N1} & \cdots & H_{NN} \end{bmatrix} \ . \tag{4}$$

We require symmetry $H^T = H$. The independent elements H_{nm} are random numbers. The random matrix H is diagonalized to obtain the energy levels $x_n,\ n = 1, 2, \ldots, N$. Indeed, a numerical simulation shows that these two models yield, after unfolding, the Poisson law and the Wigner surmise for large N, i.e. the absence or presence of correlations. This is the most important insight into the phenomenology of RMT.

In this article, we setup RMT in a more formal way, we discuss analytical calculations of correlation functions, we demonstrate how this relates to supersymmetry and stochastic field theory, we show the connection to chaos, we briefly sketch the numerous applications in many–body physics, in disordered and mesoscopic systems, in models for interacting fermions and in quantum chromodynamics. We also mention applications in other fields, even beyond physics.

Random Matrix Theory

Classical Gaussian Ensembles

For now, we consider a system whose energy levels are correlated. The $N \times N$ matrix H modeling it has no fixed zeros but random entries everywhere. There are three possible *symmetry classes* of random matrices in standard Schrödinger quantum mechanics. They are labeled by the Dyson index β. If the system is not time–reversal invariant, H has to be Hermitean and the random entries H_{nm} are complex ($\beta = 2$). If time–reversal invariance holds, two possibilities must be distinguished: If either the system is rotational symmetric, or it has integer spin and rotational symmetry is broken, the Hamilton matrix H can be chosen real symmetric ($\beta = 1$). This is the case in eqn [4]. If, on the other hand, the system has half–integer spin and rotational symmetry is broken, H is self–dual ($\beta = 4$) and the random entries H_{nm} are 2×2 quaternionic. The Dyson index β is the dimension of the number field over which H is constructed.

As we are interested in the eigenvalue correlations, we diagonalize the random matrix, $H = U^{-1} x U$. Here, $x = \mathrm{diag}\,(x_1, \ldots, x_N)$ is the diagonal matrix of the N eigenvalues. For $\beta = 4$, every eigenvalue is doubly degenerate. This is *Kramers' degeneracy*. The diagonalizing matrix U is in the orthogonal group $\mathrm{O}(N)$ for $\beta = 1$, in the unitary group $\mathrm{U}(N)$ for $\beta = 2$ and in the unitary–symplectic group $\mathrm{USp}(2N)$ for $\beta = 4$. Accordingly, the three symmetry classes are referred to as *orthogonal*, *unitary* and *symplectic*.

We have not yet chosen the probability densities for the random entries H_{nm}. To keep our assumptions about the system at a minimum, we treat all entries on equal footing. This is achieved by *rotational invariance* of the probability density $P_N^{(\beta)}(H)$, not to be confused with the rotational symmetry employed above to define the symmetry classes. No basis for the matrices is preferred in any way if we construct $P_N^{(\beta)}(H)$ from matrix invariants, i.e. from traces and determinants, such that it depends only on the eigenvalues, $P_N^{(\beta)}(H) = P_N^{(\beta)}(x)$. A particularly convenient choice is the Gaussian

$$P_N^{(\beta)}(H) = C_N^{(\beta)} \exp\left(-\frac{\beta}{4v^2} \mathrm{tr}\, H^2\right) \ , \tag{5}$$

where the constant v sets the energy scale and the constant $C_N^{(\beta)}$ ensures normalization. The three symmetry classes together with the probability densities [5] define the *Gaussian ensembles*: the Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensemble for $\beta = 1, 2, 4$.

The phenomenology of the three Gaussian ensembles differs considerably. The higher β, the stronger the level repulsion between the eigenvalues x_n. Numerical simulation quickly shows that the nearest neighbor spacing distribution behaves like $p^{(\beta)}(s) \sim s^\beta$ for small spacings s. This also becomes obvious by working out the differential probability $P_N^{(\beta)}(H) d[H]$ of the random matrices H in eigenvalue–angle coordinates x and U. Here, $d[H]$ is the invariant measure or volume element in the matrix space. When writing $d[\cdot]$, we always mean the product of all differentials of independent variables for the quantity in the square brackets. Up to constants, we have

$$d[H] = |\Delta_N(x)|^\beta d[x] d\mu(U) \ , \tag{6}$$

where $d\mu(U)$ is, apart from certain phase contributions, the invariant or Haar measure on $\mathrm{O}(N)$, $\mathrm{U}(N)$ or $\mathrm{USp}(2N)$, respectively. The Jacobian of the transformation is the modulus of the Vandermonde determinant

$$\Delta_N(x) = \prod_{n<m} (x_n - x_m) \tag{7}$$

raised to the power β. Thus, the differential probability $P_N^{(\beta)}(H) d[H]$ vanishes whenever any two eigenvalues x_n degenerate. This is the level repulsion. It immediately explains the behavior of the nearest neighbor spacing distribution for small spacings.

Additional symmetry constraints lead to new random matrix ensembles relevant in physics, the *Andreev* and the *chiral Gaussian ensembles*. If one refers to the *classical* Gaussian ensembles, one usually means the three ensembles introduced above.

Correlation Functions

The probability density to find k energy levels at positions $x_1, \ldots, x_k$ is the k–level correlation function $R_k^{(\beta)}(x_1, \ldots, x_k)$. We find it by integrating out $N-k$ levels in the N–level differential probability $P_N^{(\beta)}(H) d[H]$. We also have to average over the bases, i.e. over the diagonalizing matrices U. Due to rotational invariance, this simply yields the group volume. Thus, we have

$$R_k^{(\beta)}(x_1, \ldots, x_k) = \frac{N!}{(N-k)!} \int_{-\infty}^{+\infty} dx_{k+1} \cdots \int_{-\infty}^{+\infty} dx_N |\Delta_N(x)|^\beta P_N^{(\beta)}(x) \ . \tag{8}$$

Once more, we used rotational invariance which implies that $P_N^{(\beta)}(x)$ is invariant under permutation of the levels x_n. Since the same then also holds for the correlation functions [8], it is convenient to normalize them to the combinatorial factor in front of the integrals. A constant ensuring this has been absorbed into $P_N^{(\beta)}(x)$.

Remarkably, the integrals in eqn [8] can be done in closed form. The GUE case ($\beta = 2$) is mathematically the simplest and one finds the determinant structure

$$R_k^{(2)}(x_1, \ldots, x_k) = \det[K_N^{(2)}(x_p, x_q)]_{p,q=1,\ldots,k} \ . \tag{9}$$

All entries of the determinant can be expressed in terms of the *kernel* $K_N^{(2)}(x_p, x_q)$ which depends on two energy

arguments (x_p, x_q). Analogous but more complicated formulae are valid for the GOE ($\beta = 1$) and the GSE ($\beta = 4$), involving quaternion determinants and integrals and derivatives of the kernel.

As argued in the **Introduction**, we are interested in the energy correlations on the unfolded energy scale. The level density is formally the one–level correlation function. For the three Gaussian ensembles it is, to leading order in the level number N, the *Wigner semicircle*

$$R_1^{(\beta)}(x_1) = \frac{1}{2\pi v^2}\sqrt{4Nv^2 - x_1^2} \tag{10}$$

for $|x_1| \leq 2\sqrt{N}v$ and zero for $|x_1| > 2\sqrt{N}v$. None of the common systems in physics has such a level density. When unfolding, we also want to take the limit of infinitely many levels $N \to \infty$ to remove cutoff effects due to the finite dimension of the random matrices. It suffices to stay in the center of the semicircle where the mean level spacing is $D = 1/R_1^{(\beta)}(0) = \pi v/\sqrt{N}$. We introduce the dimensionless energies $\xi_p = x_p/D$, $p = 1, \ldots, k$ which have to be held fixed when taking the limit $N \to \infty$. The *unfolded correlation functions* are given by

$$X_k^{(\beta)}(\xi_1, \ldots, \xi_k) = \lim_{N\to\infty} D^k R_k^{(\beta)}(D\xi_1, \ldots, D\xi_k) \ . \tag{11}$$

As we are dealing with probability densities, the Jacobians $dx_p/d\xi_p$ enter the reformulation in the new energy variables. This explains the factor D^k. Unfolding makes the correlation functions *translation invariant*, they depend only on the differences $\xi_p - \xi_q$. The unfolded correlation functions can be written in a rather compact form. For the GUE ($\beta = 2$) they read

$$X_k^{(2)}(\xi_1, \ldots, \xi_k) = \det\left[\frac{\sin\pi(\xi_p - \xi_q)}{\pi(\xi_p - \xi_q)}\right]_{p,q=1,\ldots,k} \ . \tag{12}$$

There are similar, but more complicated formulae for the GOE ($\beta = 1$) and the GSE ($\beta = 4$). By construction, one has $X_1^{(\beta)}(\xi_1) = 1$.

It is useful to formulate the case where correlations are absent, i.e. the *Poisson case*, accordingly. The level density $R_1^{(\mathrm{P})}(x_1)$ is simply N times the (smooth) probability density chosen for the entries in the diagonal matrix [4]. Lack of correlations means that the k–level correlation function only involves one–level correlations,

$$R_k^{(\mathrm{P})}(x_1, \ldots, x_k) = \frac{N!}{(N-k)!N^k}\prod_{p=1}^{k} R_1^{(\mathrm{P})}(x_p) \ . \tag{13}$$

The combinatorial factor is important, since we always normalize to $N!/(N-k)!$. Hence, one finds

$$X_k^{(\mathrm{P})}(\xi_1, \ldots, \xi_k) = 1 \tag{14}$$

for all unfolded correlation functions.

Statistical Observables

The unfolded correlation functions yield all statistical observables. The two–level correlation function $X_2(r)$ with $r = \xi_1 - \xi_2$ is of particular interest in applications. If we do not write the superscript (β) or (P), we mean either of the functions. For the Gaussian ensembles, $X_2^{(\beta)}(r)$ is shown in **Figure 3**. One often writes $X_2(r) = 1 - Y_2(r)$. The two–level *cluster function* $Y_2(r)$ nicely measures the deviation from the uncorrelated Poisson case, where one has $X_2^{(\mathrm{P})}(r) = 1$ and $Y_2^{(\mathrm{P})}(r) = 0$.

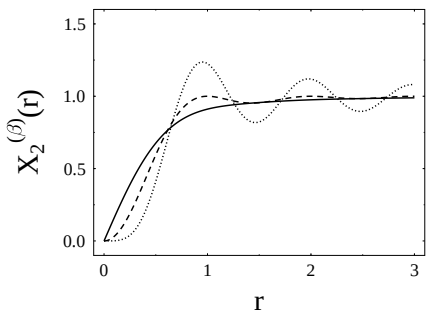

FIG. 3. Two–level correlation function $X_2^{(\beta)}(r)$ for GOE (solid), GUE (dashed) and GSE (dotted).

By construction, the average level number in an interval of length L in the unfolded spectrum is L. The *level number variance* $\Sigma^2(L)$ is shown to be an average over the two–level cluster function,

$$\Sigma^2(L) = L - 2\int_0^L (L - r)Y_2(r)dr \ . \tag{15}$$

We find $L \pm \sqrt{\Sigma^2(L)}$ levels in an interval of length L. In the uncorrelated Poisson case, one has $\Sigma^{2(\mathrm{P})}(L) = L$. This is just *Poisson's error law*. For the Gaussian ensembles $\Sigma^{2(\beta)}(L)$ behaves logarithmically for large L. The spectrum is said to be more *rigid* than in the Poisson case. As **Figure 4** shows, the level number variance probes longer distances in the spectrum, in contrast to the nearest neighbor spacing distribution.

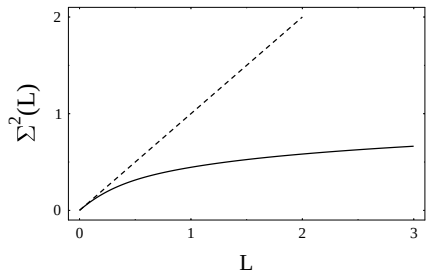

FIG. 4. Level number variance $\Sigma^2(L)$ for GOE (solid) and Poisson case (dashed).

Many more observables, also sensitive to higher order $k > 2$ correlations, have been defined. In practice, however, one is often restricted to analyzing two–level correlations. An exception is, to some extent, the nearest neighbor spacing distribution $p(s)$. It is the two–level correlation function with the *additional* requirement that the two levels in question are adjacent, i.e. that there are no levels between them. Thus, *all* correlation functions are needed if one wishes to calculate the exact nearest neighbor spacing distribution $p^{(\beta)}(s)$ for the Gaussian ensembles. These considerations explain that we have $X_2^{(\beta)}(s) \simeq p^{(\beta)}(s)$ for small s. But while $X_2^{(\beta)}(s)$ saturates for large s, $p^{(\beta)}(s)$ quickly goes to zero in a Gaussian fashion. Thus, although the nearest neighbor spacing distribution mathematically involves all correlations, it makes in practice only a meaningful statement about the two–level correlations. Luckily, $p^{(\beta)}(s)$ differs only very slightly from the heuristic Wigner surmise [2] (corresponding to $\beta = 1$), respectively from its extensions (corresponding to $\beta = 2$ and $\beta = 4$).

Ergodicity and Universality

We constructed the correlation functions as averages over an *ensemble* of random matrices. But this is not how we proceeded in the data analysis sketched in the **Introduction**. There, we started from *one single* spectrum with very many levels and obtained the statistical observable just by sampling and, if necessary, smoothing. Do these two averages, the *ensemble average* and the *spectral average*, yield the same? — Indeed, one can show that the answer is affirmative, if the level number N goes to infinity. This is referred to as *ergodicity* in RMT.

Moreover, as already briefly indicated in the **Introduction**, very many systems from different areas of physics are well described by RMT. This seems to be at odds with the Gaussian assumption [5]. There is hardly any system whose Hamilton matrix elements follow a Gaussian probability density. The solution for this puzzle lies in the unfolding. Indeed, it has been shown that almost all functional forms of the probability density $P_N^{(\beta)}(H)$ yield the same unfolded correlation functions, if no new scale comparable to the mean level spacing is present in $P_N^{(\beta)}(H)$. This is the mathematical side of the empirically found *universality*.

Ergodicity and universality are of crucial importance for the applicability of RMT in data analysis.

Wavefunctions

By modeling the Hamiltonian of a system with a random matrix H, we do not only make an assumption about the statistics of the energies, but also about those of the wavefunctions. Because of the eigenvalue equation $Hu_n = x_n u_n,\ n = 1, \ldots, N$, the wavefunction belonging to the eigenenergy x_n is modeled by the eigenvector u_n. The columns of the diagonalizing matrix $U = [u_1\ u_2\ \cdots\ u_N]$ are these eigenvectors. The probability density of the components u_{nm} of the eigenvector u_n can be calculated rather easily. For large N it approaches a Gaussian. This is equivalent to the *Porter–Thomas distribution*. While wavefunctions are often not accessible in an experiment, one can measure transition amplitudes and widths, giving information about the matrix elements of a transition operator and a projection of the wavefunctions onto a certain state in Hilbert space. If the latter are represented by a fixed matrix A or a fixed vector a, respectively, one can calculate the RMT prediction for the probability densities of the matrix elements $u_n^\dagger A u_m$ or the widths $a^\dagger u_n$ from the probability density of the eigenvectors.

Scattering systems

It is important that RMT can be used as a powerful tool in scattering theory, because the major part of the experimental information about quantum systems comes from scattering experiments. Consider an example from compound nucleus scattering. In an accelerator, a proton is shot on a nucleus, with which it forms a compound nucleus. This then decays by emitting a neutron. More generally, the ingoing *channel* ν (the proton in our example) connects to the *interaction region* (the nucleus), which also connects to an outgoing channel μ (the neutron). There are Λ channels with channel wavefunctions which are labeled $\nu = 1, \ldots, \Lambda$. The interaction region is described by an $N \times N$ Hamiltonian matrix H whose eigenvalues x_n are bound state energies labeled $n = 1, \ldots, N$. The dimension N is a cutoff which has to be taken to infinity at the end of a calculation. The $\Lambda \times \Lambda$ *scattering matrix* S contains the information about how the ingoing channels are transformed into the outgoing channels. The scattering matrix S is unitary. Under certain and often justified assumptions, a scattering matrix element can be cast into the form

$$S_{\nu\mu} = \delta_{\nu\mu} - i2\pi W_\nu^\dagger G^{-1} W_\mu \ . \tag{16}$$

The couplings $W_{n\nu}$ between the bound states n and the channels ν are collected in the $N \times \Lambda$ matrix W, W_ν is its νth column. The propagator G^{-1} is the inverse of

$$G = z1_N - H + i\pi \sum_{\nu \text{ open}} W_\nu W_\nu^\dagger \ . \tag{17}$$

Here, z is the scattering energy and the summation is only over channels which are open, i.e. accessible. Formula [16] has a clear intuitive interpretation. The scattering region is entered through channel ν, the bound states of H become *resonances* in the scattering process according to eqn [17], the interaction region is left through channel μ. This formulation applies in many areas of physics. All observables such as transmission coefficients, cross sections and others can be calculated from the scattering matrix S.

We have not made any statistical assumptions yet. Often, one can understand generic features of a scattering system by assuming that the Hamiltonian H is a random matrix, taken from one of the three classical ensembles. This is one RMT approach used in scattering theory.

Another RMT approach is based on the scattering matrix itself, S is modeled by a $\Lambda \times \Lambda$ unitary random matrix. Taking into account additional symmetries, one arrives at the three *circular ensembles*, circular orthogonal (COE), unitary (CUE) and symplectic (CSE). They correspond to the three classical Gaussian ensembles and are also labeled with the Dyson index $\beta = 1, 2, 4$. The eigenphases of the random scattering matrix correspond to the eigenvalues of the random Hamiltonian matrix. The unfolded correlation functions of the circular ensembles are identical to those of the Gaussian ensembles.

Supersymmetry

Apart from the symmetries, random matrices contain nothing but random numbers. Thus a certain type of redundancy is present in RMT. Remarkably, this redundancy can be removed, without losing any piece of information by using *supersymmetry*, i.e. by a reformulation of the random matrix model involving commuting and anticommuting variables. For the sake of simplicity, we sketch the main ideas for the GUE, but they apply to the GOE and the GSE accordingly.

One defines the k–level correlation functions by using the resolvent of the Schrödinger equation,

$$\widehat{R}_k^{(2)}(x_1, \ldots, x_k) = \frac{1}{\pi^k} \int P_N^{(2)}(H) \prod_{p=1}^{k} \mathrm{tr}\, \frac{1}{x_p^{\pm} - H} d[H] \ . \tag{18}$$

The energies carry an imaginary increment $x_p^{\pm} = x_p \pm i\varepsilon$ and the limit $\varepsilon \to 0$ has to be taken at the end of the calculation. The k–level correlation functions $R_k^{(2)}(x_1, \ldots, x_k)$ as defined in eqn [8] can always be obtained from the functions [18] by constructing a linear combination of the $\widehat{R}_k^{(2)}(x_1, \ldots, x_k)$ in which the signs of the imaginary increments are chosen such that only the imaginary parts of the traces contribute. Some trivial δ–distributions have to be removed. The k–level correlation functions [18] can be written as the k–fold derivative

$$\widehat{R}_k^{(2)}(x_1, \ldots, x_k) = \frac{1}{(2\pi)^k} \frac{\partial^k}{\prod_{p=1}^{k} \partial J_p} Z_k^{(2)}(x + J) \Big|_{J=0} \tag{19}$$

of the generating function

$$Z_k^{(2)}(x + J) = \int P_N^{(2)}(H) \prod_{p=1}^{k} \frac{\det(x_p^{\pm} + J_p - H)}{\det(x_p^{\pm} - J_p - H)} d[H] \ , \tag{20}$$

which depends on the energies and k new source variables $J_p,\ p = 1, \ldots, k$ ordered in $2k \times 2k$ diagonal matrices

$$\begin{aligned} x &= \mathrm{diag}\,(x_1, x_1, \ldots, x_k, x_k) \\ J &= \mathrm{diag}\,(+J_1, -J_1, \ldots, +J_k, -J_k) \ . \end{aligned} \tag{21}$$

We notice the normalization $Z_k^{(2)}(x) = 1$ at $J = 0$. The generating function [20] is an integral over an ordinary $N \times N$ matrix H. It can be exactly rewritten as an integral over a $2k \times 2k$ *supermatrix* σ containing commuting and anticommuting variables,

$$Z_k^{(2)}(x + J) = \int Q_k^{(2)}(\sigma) \mathrm{sdet}^{-N}(x^{\pm} + J - \sigma) d[\sigma] \ . \tag{22}$$

The integrals over the commuting variables are of the ordinary *Riemann–Stiltjes* type, while those over the anticommuting variables are *Berezin* integrals. The Gaussian probability density [5] is mapped onto its counterpart in superspace

$$Q_k^{(2)}(\sigma) = c_k^{(2)} \exp\left(-\frac{1}{2v^2} \mathrm{str}\, \sigma^2\right) \ , \tag{23}$$

where $c_k^{(2)}$ is a normalization constant. The supertrace str and the superdeterminant sdet generalize the corresponding invariants for ordinary matrices. The total number of integrations in eqn [22] is drastically reduced as compared to eqn [20]. Importantly, it is independent of the level number N which now only appears as the negative power of the superdeterminant in eqn [22], i.e. as an explicit parameter. This most convenient feature makes it possible to take the limit of infinitely many levels by means of a saddle point approximation to the generating function.

Loosely speaking, the supersymmetric formulation can be viewed as an irreducible representation of RMT which yields a clearer insight into the mathematical structures. The same is true for applications in scattering theory and in models for crossover transitions to be discussed below. This explains why supersymmetry is so often used in RMT calculations.

It should be emphasized that the rôle of supersymmetry in RMT is quite different from the one in high energy physics where the commuting and anticommuting variables represent physical particles, bosons and fermions, respectively. This is not so in the RMT context. The commuting and anticommuting variables have no direct physics interpretation, they appear simply as helpful mathematical devices to cast the RMT model into an often much more convenient form.

Crossover Transitions

The RMT models discussed up to now describe four extreme situations, the absence of correlations in the Poisson case and the presence of correlations as in the three

fully rotational invariant models GOE, GUE and GSE. A real physics system, however, is often between these extreme situations. The corresponding RMT models can vary considerably, depending on the specific situation. Nevertheless, those models in which the random matrices for two extreme situations are simply added with some weight are useful in so many applications that they acquired a rather generic standing. One writes

$$H(\alpha) = H^{(0)} + \alpha H^{(\beta)} \ , \tag{24}$$

where $H^{(0)}$ is a random matrix drawn from an ensemble with a completely arbitrary probability density $P_N^{(0)}(H^{(0)})$. The case of a fixed matrix is included, because one may choose a product of δ–distributions for the probability density. The matrix $H^{(\beta)}$ is random and drawn from the classical Gaussian ensembles with probability density $P_N^{(\beta)}(H^{(\beta)})$ for $\beta = 1, 2, 4$. One requires that the group diagonalizing $H^{(0)}$ is a subgroup of the one diagonalizing $H^{(\beta)}$. The model [24] describes a *crossover transition*. The weight α is referred to as *transition parameter*. It is useful to choose the spectral support of $H^{(0)}$ and $H^{(\beta)}$ equal. One can then view α as the root–mean–square matrix element of $H^{(\beta)}$. At $\alpha = 0$, one has the arbitrary ensemble. The Gaussian ensembles are formally recovered in the limit $\alpha \to \infty$, to be taken in a proper way such that the energies remain finite.

We are always interested in the unfolded correlation functions. Thus, α has to be measured in units of the mean level spacing D such that $\lambda = \alpha/D$ is the physically relevant transition parameter. It means that, depending on the numerical value of D, even a small effect on the original energy scale can have sizeable impact on the spectral statistics. This is referred to as *statistical enhancement.* The nearest neighbor spacing distribution is already very close to $p^{(\beta)}(s)$ for the Gaussian ensembles if λ is larger than 0.5 or so. In the long range observables such as the level number variance $\Sigma^2(L)$, the deviation from the Gaussian ensemble statistics becomes visible at interval lengths L comparable to λ.

Crossover transitions can be interpreted as *diffusion processes.* With the fictitious time $t = \alpha^2/2$, the probability density $P_N(x,t)$ of the eigenvalues x of the total Hamilton matrix $H = H(t) = H(\alpha)$ satisfies the diffusion equation

$$\Delta_x P_N(x,t) = \frac{4}{\beta}\frac{\partial}{\partial t} P_N(x,t) \ , \tag{25}$$

where the probability density for the arbitrary ensemble is the initial condition $P_N(x,0) = P_N^{(0)}(x)$. The Laplacean

$$\Delta_x = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \sum_{n<m} \frac{\beta}{x_n - x_m}\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m}\right) \tag{26}$$

lives in the curved space of the eigenvalues x. This diffusion process is *Dyson's Brownian motion* in slightly simplified form. It has a rather general meaning for *harmonic analysis on symmetric spaces*, connecting to the spherical functions of Gelfand and Harish–Chandra, Itzykson–Zuber integrals and to *Calogero–Sutherland models of interacting particles.* All this generalizes to superspace. In the supersymmetric version of Dyson's Brownian motion the generating function of the correlation functions is propagated,

$$\Delta_s Z_k(s,t) = \frac{4}{\beta}\frac{\partial}{\partial t} Z_k(s,t) \ , \tag{27}$$

where the initial condition $Z_k(s,0) = Z_k^{(0)}(s)$ is the generating function of the correlation functions for the arbitrary ensemble. Here, s denotes the eigenvalues of some supermatrices, not to be confused with the spacing between adjacent levels. Since the Laplacean Δ_s lives in this curved eigenvalue space, this diffusion process establishes an intimate connection to harmonic analysis on superspaces. Advantageously, the diffusion [27] is the same on the original and on the unfolded energy scales.

Fields of Application

Many–Body Systems

Numerous studies apply RMT to nuclear physics which is also the field of its origin. If the total number of nucleons, i.e. protons and neutrons, is not too small, nuclei show *single–particle* and *collective* motion. Roughly speaking, the former is decoherent out–of–phase motion of the nucleons confined in the nucleus, while the latter is coherent in–phase motion of all nucleons or of large groups of them such that any additional individual motion of the nucleons becomes largely irrelevant. It has been shown empirically that the single–particle excitations lead to GOE statistics, while collective excitations produce different statistics, often of the Poisson type. Mixed statistics as described by crossover transitions are then of particular interest to investigate the character of excitations. For example, one applies the model [24] with $H^{(0)}$ drawn from a Poisson ensemble and $H^{(\beta)}$ from a GOE. Another application of crossover transitions is breaking of *time–reversal invariance* in nuclei. Here, $H^{(0)}$ is from a GOE and $H^{(\beta)}$ from a GUE. Indeed, a fit of spectral data to this model yields an upper bound for the time–reversal invariance violating root–mean–square matrix element in nuclei. Yet another application is breaking of symmetries such as parity or isospin. In the case of two quantum numbers, positive and negative parity, say, one chooses $H^{(0)} = \mathrm{diag}\,(H^{(+)}, H^{(-)})$ block–diagonal with $H^{(+)}$ and $H^{(-)}$ drawn from two uncorrelated GOE and $H^{(\beta)}$ from a third uncorrelated GOE which breaks the block structure. Again, root–mean–square matrix elements for symmetry breaking have been derived from the data.

Nuclear excitation spectra are extracted from scattering experiments. An analysis as described above is only possible if the resonances are isolated. Often, this is not the case and the resonance widths are comparable to or even much larger than the mean level spacing, making it impossible to obtain the excitation energies directly from the cross sections. One then analyzes the latter and their fluctuations as measured and applies the concepts sketched above for scattering systems. This approach has also been successful for crossover transitions.

Due to the complexity of the nuclear many–body problem, one has to use effective or phenomenological interactions when calculating spectra. Hence, one often studies whether the statistical features found in the experimental data are also present in the calculated spectra which result from the various models for nuclei.

Other many–body systems, such as complex atoms and molecules, have also been studied with RMT concepts, but the main focus has always been on nuclei.

Quantum Chaos

Originally, RMT was intended for modeling systems with many degrees of freedom such as nuclei. Surprisingly, RMT proved useful for systems with few degrees of freedom as well. Most of these studied aim at establishing a link between RMT and *classical chaos*. Consider as an example the classical motion of a point–like particle in a rectangle billiard. Ideal reflection at the boundaries and absence of friction are assumed, implying that the particle is reflected infinitely many times. A second billiard is built by taking a rectangle and replacing one corner with a quarter circle as shown in **Figure 5**. The motion of the particle in this *Sinai billiard* is very different from the one in the rectangle. The quarter circle acts like a convex mirror which spreads out the rays of light upon reflection. This effect accumulates, because the vast majority of the possible trajectories hit the quarter circle infinitely many times under different angles. This makes the motion in the Sinai billiard *classically chaotic*, while the one in the rectangle is *classically regular*. The rectangle is *separable* and *integrable*, while this feature is destroyed in the Sinai billiard. One now quantizes these billiard systems, calculates the spectra and analyzes their statistics. Up to certain scales, the rectangle (for irrational squared ratio of the side lengths) shows Poisson behavior, the Sinai billiard yields GOE statistics.

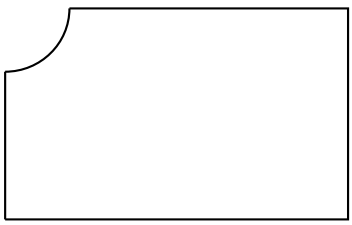

FIG. 5. The Sinai billiard.

A wealth of such empirical studies led to the Bohigas–Giannoni–Schmit conjecture. We state it here not in its original, but in a frequently used form: *Spectra of systems whose classical analogues are fully chaotic show correlation properties as modeled by the Gaussian ensembles.* The Berry–Tabor conjecture is complementary: *Spectra of systems whose classical analogues are fully regular show correlation properties which are often those of the Poisson type.* As far as concrete physics applications are concerned, these conjectures are well–posed. From a strict mathematical viewpoint, they have to be supplemented with certain conditions to exclude exceptions such as *Artin's billiard*. Due to the definition of this system on the hyperbolic plane, its quantum version shows Poisson–like statistics, although the classical dynamics is chaotic. Up to now, no general and mathematically rigorous proofs of the conjectures could be given. However, *semiclassical* reasoning involving *periodic orbit theory* and, in particular, the *Gutzwiller trace formula*, yields at least a heuristic understanding.

Quantum chaos has been studied in numerous systems. An especially prominent example is the Hydrogen atom put in a strong magnetic field which breaks the integrability and drives the correlations towards the GOE limit.

Disordered and Mesoscopic Systems

An electron moving in a probe, a piece of wire, say, is many times scattered at impurities in the material. This renders the motion diffusive. In a statistical model, one writes the Hamilton operator as a sum of the kinetic part, i.e. the Laplacean, and a white–noise *disorder potential* $V(\vec{r})$ with second moment

$$\langle V(\vec{r})V(\vec{r}\,')\rangle = c_V \delta^{(d)}(\vec{r}-\vec{r}\,') \ . \tag{28}$$

Here, $\vec{r}$ is the position vector in d dimensions. The constant c_V determines the mean free time between two scattering processes in relation to the density of states. It is assumed that phase coherence is present such that quantum effects are still significant. This defines the *mesoscopic regime*. The average over the disorder potential can be done with supersymmetry. In fact, this is the context in which supersymmetric techniques in statistical physics were developed, before they were applied to RMT models. In the case of weak disorder, the resulting field theory in superspace for two–level correlations acquires the form

$$\int d\mu(Q) f(Q) \exp\left(-S(Q)\right) \tag{29}$$

where $f(Q)$ projects out the observable under consideration and where $S(Q)$ is the *effective Lagrangean*

$$S(Q) = -\int \mathrm{str}\,\left(\mathcal{D}(\nabla Q(\vec{r}))^2 + i2rMQ(\vec{r})\right) d^d r \ . \tag{30}$$

This is the *supersymmetric non–linear σ model*. It is used to study level correlations, but also to obtain information about the conductance and conductance fluctuations when the probe is coupled to external leads. The supermatrix field $Q(\vec{r})$ is the remainder of the disorder average, its matrix dimension is four or eight, depending on the symmetry class. This field is a Goldstone mode. It does not directly represent a particle as often the case in high energy physics. The matrix $Q(\vec{r})$ lives in a coset space of certain supergroups. A tensor M appears in the calculation, and r is the energy difference on the unfolded scale, not to be confused with the position vector $\vec{r}$.

The first term in the effective Lagrangean involving a gradient squared is the *kinetic term*, it stems from the Laplacean in the Hamiltonian. The constant $\mathcal{D}$ is the classical diffusion constant for the motion of the electron through the probe. The second term is the *ergodic term*. In the limit of zero dimensions, $d \to 0$, the kinetic term vanishes and the remaining ergodic term yields precisely the unfolded two–level correlations of the Gaussian ensembles. Thus, RMT can be viewed as the zero–dimensional limit of field theory for disordered systems. For $d > 0$, there is a competition between the two terms. The diffusion constant $\mathcal{D}$ and the system size determine an energy scale, the *Thouless energy* E_c, within which the spectral statistics is of the Gaussian ensemble type and beyond which it approaches the Poisson limit. In **Figure 6**, this is schematically shown for the level number variance $\Sigma^2(L)$ which bends from Gaussian ensemble to Poisson behavior when $L > E_c$. This relates to the crossover transitions in RMT. Gaussian ensemble statistics means that the electron states extend over the probe, while Poisson statistics implies their spatial localization. Hence, the Thouless energy is directly the dimensionless conductance.

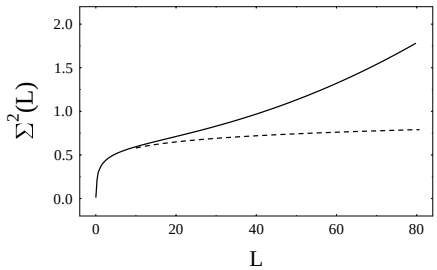

FIG. 6. Level number variance $\Sigma^2(L)$. In this example, the Thouless energy is $E_c \approx 10$ on the unfolded scale. The Gaussian ensemble behavior is dashed.

A large number of issues in disordered and mesoscopic systems has been studied with the supersymmetric non–linear σ model. Most results have been derived for *quasi–one–dimensional* systems. Through a proper discretization, a link is established to models involving chains of random matrices. As the conductance can be formulated in terms of the scattering matrix, the experience with RMT for scattering systems could be applied and led to numerous new results.

Quantum Chromodynamics

Quarks interact by exchanging gluons. In quantum chromodynamics, the gluons are described by gauge fields. Relativistic quantum mechanics has to be used. Analytical calculations are only possible after some drastic assumptions and one must resort to *lattice gauge theory*, i.e. to demanding numerics, to study the full problem.

The massless Dirac operator has *chiral symmetry*, implying that all non–zero eigenvalues come in pairs $(-\lambda_n, +\lambda_n)$ symmetrically around zero. In *chiral* RMT, the Dirac operator is replaced with block off–diagonal matrices

$$W = \begin{bmatrix} 0 & W_b \\ W_b^\dagger & 0 \end{bmatrix} , \tag{31}$$

where W_b is a random matrix without further symmetries. By construction, W has chiral symmetry. The assumption underlying chiral RMT is that the gauge fields effectively randomize the motion of the quark. Indeed, this simple schematic model correctly reproduces low–energy sum rules and spectral statistics of lattice gauge calculations. Near the center of the spectrum, there is a direct connection to the partition function of quantum chromodynamics. Furthermore, a similarity to disordered systems exists and an analogue of the Thouless energy could be found.

Other Fields

Of the wealth of further investigations, we can mention but a few. RMT is in general useful for wave phenomena of all kinds, including classical ones. This has been shown for *elastomechanical* and *electromagnetic* resonances.

An important field of application is *quantum gravity* and matrix model aspects of *string theory*. We decided not to go into this, because the reason for the emergence of RMT concepts there is very different from everything else discussed above.

RMT is also successful beyond physics. Not surprisingly, it always received interest in mathematical statistics, but, as already said, it also relates to harmonic analysis. A connection to number theory exists as well. The high–lying zeros of the Riemann ζ function follow the GUE predictions over certain interval lengths. Unfortunately, a deeper understanding is still lacking.

As the interest in statistical concepts grows, RMT keeps finding new applications. Recently, one even started using RMT for risk management in finance.

See also

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390 — Quantum mechanics
453 — Quantum mechanical scattering theory

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31 — Integrable systems in random matrix theory
68 — Symmetry classes in random matrix theory
279 — Supermanifolds
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406 — Hyperbolic billiards
449 — Arithmetic quantum chaos
65 — Dynamics in disordered systems
63 — Random walks in random environments
293 — Quantum chromodynamics
435 — Random wave models
30 — Growth processes in random matrix theory

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